Algebraic Structures on Grothendieck Groups

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Table of Contents

Preface

These notes were completed in the summer of 2013 under the supervision of Prof. Alistair Savage as part of a Work-Study Program. This research was supported by the University of Ottawa and the NSERC Discovery Grant of Prof. Savage. The purpose of the document is to serve as an introductory reference for students who have taken advanced undergraduate courses in algebra (e.g. MAT3141 – Linear Algebra II and MAT3143 – Ring Theory at the University of Ottawa) and are interested in the field of algebraic (de)categorification.

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Introduction

Categorification, a term introduced by Louis Crane and Igor Frenkel, is the process of replacing set-theoretic notions by their corresponding category-theoretic analogues. We replace sets with categories, elements with objects, functions with functors, and equations with natural isomorphisms. The goal of categorification is to obtain extra structure on the original object with which to study it. However, the opposite process, *decategorification*, wherein isomorphic objects are identified as "equal", is a more natural starting point of study as it is easier to forget information than to create it.

Consider, for instance, a scenario[1] set somewhere around the dawn of civilization where a horse tamer wishes to gift two of his patrons an equal number of horses. He picks out the horses to give away, and divides them into two groups. He then tries to verify that he is indeed being fair to his patrons, attempting to construct an isomorphism by lining up each horse in one group to a horse in the other. This, however, turns out to be quite a time-consuming and confusing process since the horses refuse to stay still and reins hadn't yet been invented. But then, he thinks of a solution! Instead of trying to find an explicit isomorphism between these two finite sets, he uses some abstract "counting system" consisting of "1", "2", "3", ..., to "count" each group! No longer concerned with the fact that he is dealing with horses at all, he has made the task easier by only remembering the necessary information. And thus, by decategorifying the category of finite sets, we obtain the natural numbers $\mathbb{N}$.

The first chapter of this paper provides the reader with the necessary background knowledge in category theory – and of course, readers with prior knowledge of the subject may wish to skip straight to Chapter 2. It is in Chapter 2 that we discuss one of the most natural ways to decategorify the information in a category – that of taking its corresponding *Grothendieck group*[2]. Certain types of categories (*additive, abelian* and *triangulated* categories) lend themselves to the construction of Grothendieck groups in a natural way. We discuss properties that these categories can have that give the Grothendieck groups nice bases. Finally, in Chapter 3, we discuss categories with extra information (*monoidal* categories) whose Grothendieck groups (sometimes) have the structure of a ring, and conclude by relating abelian categories to modules over a ring.

[1]This scenario is purely hypothetical. The authors make no claim as to whether the invention of $\mathbb{N}$ can actually be attributed to a horse tamer.

[2]The Grothendieck group is named for Alexander Grothendieck, author of the modern mathematical masterpiece *Éléments de Géométrie Algébrique.*

Chapter 1

Categories

In this chapter, we first develop the definition of a category and provide examples of familiar algebraic structures viewed as categories. Then, we examine some special objects and morphisms that often exist in categories. Finally, we consider categories themselves as entities with morphisms from one to the other, in the form of functors and natural transformations.

1.1 The Notion of a Category

Before we can give the definition of a category, we require some set theory. We would like to define a structure that contains all sets, which begins with the following observation:

Proposition 1.1.1. *There is no set which contains every set.*

Proof. Suppose, on the contrary, that there exists some set X such that X is the set of all sets. We know that for any set, there exists a subset consisting of elements of that set that satisfy any given property. So there is a subset $A = \{x \in X \mid x \notin x\} \subseteq X$. Since A is a set, it is a member of X. Now if $A \in A$, then $A \notin A$. But if $A \notin A$, then $A \in A$. These two cases are exhaustive and in both, we have $A \in A$ and $A \notin A$. This is, of course, a load of nonsense. Thus, no such set X exists. This well-known result is called *Russell's Paradox*. □

Therefore, we need to define a new structure that is "large enough" to contain all sets.

Definition 1.1.2 (Class)**.** A *class* is a collection of objects for which the following are true:

(a) All members of a class are sets.

(b) For any property P, there exists the *class of all sets with property P*.

(c) If $C_1, ..., C_n$ are classes, then the n-tuple $(C_1, ..., C_n)$ is a class.

(d) All sets are classes (and hence, all members of sets are sets).

(e) The largest class is called the *universe*, which we denote by $\mathcal{U}$; it is precisely the class of all sets.

In practice, the things we know about sets (subsets, unions, intersections, cartesian products, relations and functions) behave exactly the same way for classes. If a class is a set, then we call it a *small class*. Otherwise, we say that the class is a *proper class* or *large class*. Since a proper class is not a set, no proper class can be a member of a class. If we now revisit Russell's Paradox by considering $\mathcal{U}$ as the class of all sets, then the subclass $A = \{x \in \mathcal{U} \mid x \notin x\} \subseteq \mathcal{U}$ does not cause any problems! The subclass A is simply the class of all sets which are not members of themselves, and is a proper class.

Remark 1.1.3. A set can be viewed as a class by identifying each element with the set containing precisely that element.

Definition 1.1.4 (Category)**.** A *category* consists of a class Ob of *objects* and a class Mor of *morphisms* (also called *arrows*) together with four assignments called domain, codomain, identity and composition:

(a) Domain (respectively, codomain) is an assignment of the form Mor $\to$ Ob that assigns to every morphism f an object $\mathrm{dom}(f)$ (respectively, $\mathrm{cod}(f)$). We write

$$f\colon a \to b \quad \text{or} \quad a \xrightarrow{f} b$$

to mean that a morphism f satisfies $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$.

(b) Identity is an assignment of the form Ob $\to$ Mor that assigns to every object a a morphism id_a such that $\mathrm{cod}(\mathrm{id}_a) = a = \mathrm{dom}(\mathrm{id}_a)$.

(c) We define the class K of *composable morphisms* to be a subclass of $\mathrm{Mor} \times \mathrm{Mor}$ that consists of all pairs (g, f) such that $\mathrm{dom}(g) = \mathrm{cod}(f)$. Composition is an assignment of the form $K \to$ Mor that assigns to every pair (g, f) of composable morphisms a new morphism $g \circ f$ such that $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$. The morphism $g \circ f$ is called the *composite* morphism of g and f.

In addition, the following axioms must be satisfied:

(CA1) *Associativity*: for any sequence of objects and morphisms of the form $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(CA2) *Unit Law*: for any morphism $f\colon a \to b$, we have $\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a$.

When working with one or more categories in a specific context, we will write $\mathrm{Ob}\,\mathbf{C}$ and $\mathrm{Mor}\,\mathbf{C}$ to respectively mean the objects and morphisms of the category $\mathbf{C}$. For brevity, we will sometimes use a common abuse of notation and write $c \in \mathbf{C}$ for $c \in \mathrm{Ob}\,\mathbf{C}$.

Definition 1.1.5 (Small category, large category)**.** We say a category $\mathbf{C}$ is a *small category* if $\mathrm{Ob}\,\mathbf{C}$ and $\mathrm{Mor}\,\mathbf{C}$ are sets. Otherwise, $\mathbf{C}$ is a *large category* .

Definition 1.1.6 (Homomorphism class, locally small category)**.** Let $\mathbf{C}$ be a category. We define the *homomorphism class from a to b* to be

$$\mathrm{Hom}_{\mathbf{C}}(a,b) := \{f \in \mathrm{Mor} \mid \mathrm{dom}(f) = a \text{ and } \mathrm{cod}(f) = b\}.$$

We will often say *hom-class* for short and write $\mathrm{Hom}(a,b)$ if the category is understood. A category whose hom-classes are, in fact, sets is called a *locally small category*.

Remark 1.1.7. The term *hom-set* appears in place of hom-class in some of the literature (for instance, [15]), where different definitions of small and large categories are given. Since for us, a hom-class is not necessarily a set, we will refrain from using the term hom-set.

Definition 1.1.8 (Subcategory, full subcategory)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then, $\mathbf{D}$ is a *subcategory* of $\mathbf{C}$ if the following hold:

(a) $\mathrm{Ob}\,\mathbf{D} \subseteq \mathrm{Ob}\,\mathbf{C}$.

(b) $\mathrm{Mor}\,\mathbf{D} \subseteq \{f \in \mathrm{Mor}\,\mathbf{C} \mid \mathrm{dom}(f), \mathrm{cod}(f) \in \mathrm{Ob}\,\mathbf{D}\}$.

(c) For all $a \in \mathrm{Ob}\,\mathbf{D}$, we have $\mathrm{id}_a \in \mathrm{Mor}\,\mathbf{D}$.

(d) If $f, g \in \mathrm{Mor}\,\mathbf{D}$ and $\mathrm{dom}(g) = \mathrm{cod}(f)$, then $g \circ f \in \mathrm{Mor}\,\mathbf{D}$.

(e) Composition in $\mathbf{D}$ is given by composition in $\mathbf{C}$.

Additionally, if $\mathrm{Hom}_{\mathbf{D}}(a,b) = \mathrm{Hom}_{\mathbf{C}}(a,b)$ for all pairs (a,b) of objects in $\mathbf{D}$, then $\mathbf{D}$ is a *full subcategory* of $\mathbf{C}$.

Examples 1.1.9. The following are examples of small categories:

(a) The empty category $\mathbf{0}$ has $\mathrm{Ob} = \varnothing$ and $\mathrm{Mor} = \varnothing$.

(b) The category $\mathbf{1}$ has $\mathrm{Ob} = \{a\}$ and $\mathrm{Mor} = \{\mathrm{id}_a\}$.

(c) The category $\mathbf{2}$ has $\mathrm{Ob} = \{a, b\}$ and $\mathrm{Mor} = \{\mathrm{id}_a, \mathrm{id}_b, a \xrightarrow{f} b\}$.

(d) The category $\mathbf{3}$ has $\mathrm{Ob} = \{a, b, c\}$ and the three identity morphisms along with three non-identity morphisms arranged in a triangle as follows:

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c \arrow[ur] & \end{tikzcd}$$

(e) The category $\downdownarrows$ has $\mathrm{Ob} = \{a, b\}$ and $\mathrm{Mor} = \{\mathrm{id}_a, \mathrm{id}_b, a \xrightarrow{f} b, a \xrightarrow{g} b\}$. We call f and g *parallel morphisms*.

Examples 1.1.10. The following are examples of large categories:

(a) **Set** is the category of sets, where the objects are all sets and the morphisms are all functions between sets.

(b) **FinSet** is the category of finite sets, a full subcategory of **Set**.

(c) **Grp** is the category of groups, where the objects are all groups and the morphisms are all group homomorphisms.

(d) **Ab** is the category of abelian groups, a full subcategory of **Grp**.

(e) $\mathbf{Ab}^{\text{fg}}$ is the category of finitely generated abelian groups, a full subcategory of **Ab**.

(f) **FinAb** is the category of finite abelian groups, a full subcategory of $\mathbf{Ab}^{\text{fg}}$.

(g) **Mon** is the category of monoids, where the objects are all monoids and the morphisms are all monoid homomorphisms.

(h) **Ring** is the category of rings, where the objects are all rings and the morphisms are all (unity-preserving) ring homomorphisms.

(i) **CRing** is the category of commutative rings, a full subcategory of **Ring**.

(j) Given a ring R, the category R-**Mod** has all left modules over R as its objects and all module homomorphisms between them as its morphisms.

(k) Given a field $\mathbb{K}$, the category $\mathbf{Vect}_{\mathbb{K}}$ has all vector spaces over $\mathbb{K}$ as its objects and all $\mathbb{K}$-linear transformations as its morphisms.

(l) Given a field $\mathbb{K}$, the category $\mathbf{FinVect}_{\mathbb{K}}$ is the full subcategory of $\mathbf{Vect}_{\mathbb{K}}$ consisting of all finite-dimensional vector spaces over $\mathbb{K}$.

Example 1.1.11. The category **Rel** has sets as its objects and binary relations as its morphisms. If R and S are composable relations, then the composite relation is defined by

$$(a, c) \in S \circ R \text{ if and only if there exists } b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S.$$

For any object A, the identity morphism is the relation $\mathrm{id}_A = \{(a, a) \mid a \in A\}$.

Example 1.1.12. A *discrete category* is category such that every morphism is an identity morphism (e.g. **1** is the discrete category with a single object). Note that every discrete category is uniquely determined by its class of objects and, furthermore, every class X defines the object class of a discrete category with $\mathrm{Mor} = \{\mathrm{id}_x \mid x \in X\}$. A category **C** is discrete if and only if every subcategory of **C** is a full subcategory.

Example 1.1.13. Consider a small category $\mathbf{C}$ where $\mathrm{Ob} = \{a\}$ but where there may be any number of nonidentity morphisms. Since the only object is a, any pair of morphisms f, g satisfy $\mathrm{cod}(f) = a = \mathrm{dom}(g)$ and so are composable. By CA1, composition is associative and by CA2, the identity morphism acts as a left and right identity for composition. Hence, $\mathrm{Mor}\,\mathbf{C}$ is a monoid under composition. Indeed, given any object x in any small category, the set $\mathrm{Hom}(x, x)$ is a monoid under composition.

Example 1.1.14. Let V be a class (recall that $X \in V \implies X$ is a set). Then, $\mathbf{Ens}_V$ is a category[1] where $\mathrm{Ob} = V$ and $\mathrm{Hom}(X, Y) = \{f \mid f \text{ is a function } X \to Y\}$ for each pair (X, Y) of sets in V. The identity morphisms are simply the identity functions and composition of functions is interpreted in the usual way.

Definition 1.1.15 (Opposite category)**.** For any category $\mathbf{C}$, we define its *opposite category* $\mathbf{C}^{\mathrm{op}}$ to be the category with $\mathrm{Ob}\,\mathbf{C}^{\mathrm{op}} = \mathrm{Ob}\,\mathbf{C}$ and $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A)$ for every pair (A, B) of objects – that is, for every morphism $f\colon A \to B \in \mathrm{Mor}\,\mathbf{C}$ we have the corresponding morphism $f^{\mathrm{op}}\colon B \to A \in \mathrm{Mor}\,\mathbf{C}^{\mathrm{op}}$. Composition of morphisms in $\mathbf{C}^{\mathrm{op}}$ is given by $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$ and is defined precisely when $g \circ f$ is defined in $\mathbf{C}$.

Definition 1.1.16 (Product category)**.** Given two categories $\mathbf{C}$ and $\mathbf{D}$, we define the *product category* $\mathbf{C} \times \mathbf{D}$ to be the category whose objects are pairs (c, d) with $c \in \mathbf{C}$ and $d \in \mathbf{D}$, and whose morphisms $(c, d) \to (c', d')$ are pairs (f, g) of morphisms $c \xrightarrow{f} c'$ and $d \xrightarrow{g} d'$. The identity morphism of (c, d) is simply $(\mathrm{id}_c, \mathrm{id}_d)$ and composition of two morphisms

$$(c, d) \xrightarrow{(f,g)} (c', d') \xrightarrow{(f',g')} (c'', d'')$$

is given by

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g).$$

Definition 1.1.17 (Morphism category)**.** Let $\mathbf{C}$ be a category. The *morphism category* of $\mathbf{C}$, denoted $\mathbf{C}^{\to}$, is the category given by the following data:

(a) $\mathrm{Ob}\,\mathbf{C}^{\to} = \mathrm{Mor}\,\mathbf{C}$

(b) $\mathrm{Hom}_{\mathbf{C}^{\to}}(f, f') = \{(g, g') \mid g, g' \in \mathrm{Mor}\,\mathbf{C} \text{ and } f' \circ g = g' \circ f\}$

(c) For any morphism $f\colon a \to b$, $\mathrm{id}_f = (\mathrm{id}_a, \mathrm{id}_b)$.

(d) Composition is given by $(g, g') \circ (h, h') = (g \circ h, g' \circ h')$.

In other words, for any objects $f\colon a \to b$ and $f'\colon a' \to b'$ in $\mathbf{C}^{\to}$, a morphism $(g, g')\colon f \to f'$ in $\mathbf{C}^{\to}$ consists of a pair of morphisms $g\colon a \to a'$ and $g'\colon b \to b'$ in $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "g'"] \\ a' \arrow[r, "f'"'] & b' \end{tikzcd}$$

[1]This notation, which appears in [15], is likely to originate from *ensemble*, the French word for “set”.

1.2 Objects & Morphisms

Now that we have plenty of categories to work with, we turn our attention to special objects and morphisms that may exist in them. Many of these notions are similar to those in group theory and ring theory, but there are sometimes subtle differences. One should be careful not to simply gloss over a definition because it shares a name with a similar term in a different context!

Definition 1.2.1 (Isomorphism, isomorphic objects)**.** A morphism $f\colon a \to b$ is an *isomorphism* if there exists a morphism $g\colon b \to a$ such that $g \circ f = \mathrm{id}_a$ and $f \circ g = \mathrm{id}_b$. We say that g is the *inverse* of f and write $g = f^{-1}$.

For any pair of objects (a, b), if an isomorphism $f\colon a \to b$ exists, we say that a is *isomorphic* to b and write $a \cong b$.

Proposition 1.2.2. *In any category we have the following:*

(a) The composition of two isomorphisms is an isomorphism.

(b) Being isomorphic is an equivalence relation on the objects.

Proof. (a) Let $f\colon a \to b$ and $g\colon b \to c$ be isomorphisms. Then, we have $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_a$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_c$ so $g \circ f$ is an isomorphism.

(b) Reflexivity is obvious (for any object a, id_a is an isomorphism), and symmetry follows directly from the definition. Suppose now that $a \cong b$ and $b \cong c$. Then, there exist isomorphisms $f\colon a \to b$ and $g\colon b \to c$. By (a), we have $g \circ f\colon a \to c$ is an isomorphism, so $a \cong c$, proving transitivity . Therefore, the relation $\cong$ is an equivalence relation on the objects. □

Remark 1.2.3. We say two morphisms f and g are isomorphic if they are isomorphic as objects in the morphism category and write $f \cong g$. Equivalently, we have $f \cong g$ if and only if there exist two isomorphisms h and h' such that $f = h \circ g \circ h'$.

In this paper, we will discuss several objects defined by a property that are *unique up to isomorphism*. By this, we mean that all objects fulfilling the property are isomorphic. Furthermore, all properties studied in this paper will be *invariant under isomorphism* — that is, if $a \cong b$, then a has the property if and only if b has the property. Thus, any object that is unique up to isomorphism will be characterized by all the objects contained in precisely one isomorphism equivalence class.

Definition 1.2.4 (Skeleton subcategory, essentially small category)**.** A category $\mathbf{D}$ is a *skeleton subcategory* of $\mathbf{C}$ if it has the following properties:

(a) The category $\mathbf{D}$ is a full subcategory of $\mathbf{C}$.

(b) Every object in $\mathbf{C}$ is isomorphic to some object in $\mathbf{D}$.

(c) No two distinct objects in $\mathbf{D}$ are isomorphic.

We will sometimes refer to a skeleton subcategory of $\mathbf{C}$ as simply a skeleton of $\mathbf{C}$ and denote it by $\underline{\mathbf{C}}$. A category is *essentially small* if it has a small skeleton.

Remark 1.2.5. Essentially small categories are locally small.

Example 1.2.6. (a) The category $\mathbf{FinVect}_{\mathbb{C}}$ of finite-dimensional vector spaces over the complex numbers has a skeleton subcategory with objects $\{\mathbb{C}^n \mid n \in \mathbb{N}\}$.

(b) The category $\mathbf{Ab}^{\mathrm{fg}}$ of finitely generated abelian groups has a skeleton subcategory with

$$\operatorname{Ob} \underline{\mathbf{Ab}}^{\mathrm{fg}} = \{\mathbb{Z}^k \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m} \mid n_i | n_{i+1} \text{ and } k, m \in \mathbb{N}\}.$$

See [19, p. 338] for the group theory details.

Definition 1.2.7 (Initial/terminal/zero object)**.** Let $\mathbf{C}$ be a category. We say that $a \in \mathbf{C}$ is an *initial object* if for every object $c \in \mathbf{C}$, there exists a unique morphism $a \to c$. Conversely, $b \in \mathbf{C}$ is a *terminal object* if for every object $c \in \mathbf{C}$, there exists a unique morphism $c \to b$. A *zero object* is an object that is both initial and terminal.

Remark 1.2.8. Note that an initial object in $\mathbf{C}$ is a terminal object in the opposite category $\mathbf{C}^{\mathrm{op}}$, and vice versa. Hence, we say that initial and terminal objects are *dual* notions to each other. When a certain type of object or morphism is dual to another, the details of proofs involving them are often similar – one simply reverses the direction of the morphisms involved. For this reason, we sometimes prove only one of the dual cases and remark that the details of the other are analogous.

Lemma 1.2.9. *The initial, terminal, and zero objects of a category, if they exist, are unique up to isomorphism.*

Proof. Suppose a and a' are initial objects of a category $\mathbf{C}$. Then there exists a unique morphism $f\colon a \to a'$ and a unique morphism $g\colon a' \to a$. Then, we have the composite morphisms $g \circ f\colon a \to a$ and $f \circ g\colon a' \to a'$. But then, since there is exactly one morphism $a \to a$ and exactly one morphism $a' \to a'$, these composites must be the identity morphisms of a and a' respectively. Hence, $a \cong a'$. The proof for terminal objects is analogous, and the result for zero objects follows directly. □

Examples 1.2.10. (a) A discrete category with more than one object has neither an initial object nor a terminal object.

(b) In **Set**, the initial object is $\varnothing$ since for any set X, there is exactly one function $\varnothing \to X$. The terminal object is $\{\star\}$, a set containing one element, since for any set X there is exactly one function $X \to \{\star\}$.

(c) In **Grp** and **Ab**, the zero object is the trivial group $\{\varepsilon\}$.

(d) In **Rel**, the zero object is $\varnothing$ since any relation that has $\varnothing$ as its domain or codomain must be the empty relation.

Definition 1.2.11 (Constant/coconstant/zero morphism)**.** A morphism $f\colon a \to b$ is called a *constant morphism* if for any object c and morphisms $g\colon c \to a$ and $h\colon c \to a$ we have $f \circ g = f \circ h$. A morphism $f\colon a \to b$ is called a *coconstant morphism* if for any object c and morphisms $g\colon b \to c$ and $h\colon b \to c$ we have $g \circ f = h \circ f$. A morphism that is both constant and coconstant is called a *zero morphism*. We say a category *has zero morphisms* if for every pair of objects (a, b), there exists a zero morphism, denoted $0_{a,b}\colon a \to b$.

Examples 1.2.12. (a) Constant morphisms in **Set** are precisely constant functions. Furthermore, constant functions in any subcategory of **Set** are constant morphisms, but there may be nonconstant functions that are also constant morphisms. For instance, in the discrete subcategory of **Set**, all identity morphisms are constant.

(b) In **Set**, the only coconstant morphisms are the empty functions $f_X\colon \varnothing \to X$ for any set X. Hence, for any set $A \neq \varnothing$ and any set B, a zero morphism $0_{A,B}$ does not exist. Thus, the category **Set** does not have zero morphisms.

(c) For any $G, H \in$ **Grp**, the trivial homomorphism $\phi\colon G \to H$ which sends all elements of G to the identity element of H is the zero morphism $0_{G,H}$. Thus, the category **Grp** has zero morphisms.

Proposition 1.2.13. *If a category has zero morphisms, then we have the following:*

(a) For every pair of objects (a, b), the morphism $0_{a,b}$ is unique.

(b) For any morphisms $f\colon a \to b$ and $g\colon b \to c$, we have $0_{b,c} \circ f = 0_{a,c} = g \circ 0_{a,b}$.

Proof. (a) Suppose $0'_{a,b}\colon a \to b$ is also a zero morphism. Then, we have $0'_{a,b} = 0'_{a,b} \circ \mathrm{id}_a = 0'_{a,b} \circ 0_{a,a} = 0_{a,b} \circ 0_{a,a} = 0_{a,b} \circ \mathrm{id}_a = 0_{a,b}$.

(b) By definition, $0_{b,c} \circ f = 0_{b,c} \circ 0_{a,b}$. We leave to the reader the verification that the composition of two zero morphisms is again a zero morphism. By the uniqueness of zero morphisms, we have $0_{b,c} \circ 0_{a,b} = 0_{a,c}$ and so $0_{b,c} \circ f = 0_{a,c}$, as desired. Analogously, $g \circ 0_{a,b} = 0_{a,c}$. □

Lemma 1.2.14. *If a category* **C** *has a zero object* 0*, then it has zero morphisms.*

Proof. Let $a, b \in$ **C**. By the definition of the zero object, there exist morphisms $f\colon a \to 0$ and $g\colon 0 \to b$. We claim that $g \circ f\colon a \to b$ is a zero morphism.

Suppose we have morphisms $h\colon c \to a$ and $k\colon c \to a$. Since 0 is terminal, we have $f \circ h = f \circ k$ and hence, $(g \circ f) \circ h = (g \circ f) \circ k$. Thus, $g \circ f$ is constant.

The proof that $g \circ f$ is coconstant uses the fact that 0 is initial and is similar. Therefore, $g \circ f$ is a zero morphism and thus, **C** has zero morphisms. □

Definition 1.2.15 (Monomorphism, epimorphism, bimorphism, balanced category)**.**
A *monomorphism* is a morphism $m\colon b \to c$ such that for any two morphisms $f\colon a \to b$ and $g\colon a \to b$ we have

$$m \circ f = m \circ g \implies f = g.$$

An *epimorphism* is a morphism $e\colon a \to b$ such that for any two morphisms $f\colon b \to c$ and $g\colon b \to c$ we have

$$f \circ e = g \circ e \implies f = g.$$

We also say that a morphism is *monic* if it is a monomorphism or *epic* if it as an epimorphism. A morphism which is both monic and epic is called a *bimorphism* . A category where all bimorphisms are isomorphisms is called *balanced*.

Remark 1.2.16. The notions of monomorphisms and epimorphisms are dual to each other. Every monomorphism in $\mathbf{C}$ is an epimorphism in $\mathbf{C}^{\mathrm{op}}$, and vice versa.

Remark 1.2.17. Categories can have morphisms that are not functions so in general, it does not make sense to ask if a morphism is injective or surjective. If our morphisms are functions with the usual composition, then any injective morphism is a monomorphism and any surjective morphism is an epimorphism. The converse of these statements are not true. Consider the following functions:

$$\begin{aligned} f\colon \{1,2\} &\to \{1\} \\ 1 &\mapsto 1 \\ 2 &\mapsto 1 \end{aligned} \qquad\qquad \begin{aligned} g\colon \{1\} &\to \{1,2\} \\ 1 &\mapsto 1 \end{aligned}$$

With the usual identity functions and the usual composition, it is easily verified that $\mathbf{C}$ and $\mathbf{D}$ are categories wherein $\operatorname{Ob}\mathbf{C} = \operatorname{Ob}\mathbf{D} = \{\{1\},\{1,2\}\}$, $\operatorname{Mor}\mathbf{C} = \{\mathrm{id}_{\{1\}}, \mathrm{id}_{\{1,2\}}, f\}$, and $\operatorname{Mor}\mathbf{D} = \{\mathrm{id}_{\{1\}}, \mathrm{id}_{\{1,2\}}, g\}$. Indeed, f is a monomorphism in $\mathbf{C}$ (but not injective) and g is a epimorphism in $\mathbf{D}$ (but not surjective).

Example 1.2.18. **Set**, **Rel**, and **Grp** are balanced categories. The category **Ring** is not balanced. For example, we shall show that the inclusion function $i\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ given by $i(x) = \frac{x}{1}$ is a bimorphism that is not an isomorphism. Since i is injective, it is a monomorphism by Remark 1.2.17.

Suppose now that for some ring R, there exist ring homomorphisms $g\colon \mathbb{Q} \to R$ and $h\colon \mathbb{Q} \to R$ such that $g \circ i = h \circ i$. Then, $g(\frac{x}{1}) = h(\frac{x}{1})$ for all $x \in \mathbb{Z}$. Furthermore, for any integer x and nonzero integer y we have the following, recalling that the multiplicative inverse commutes with ring homomorphisms:

$$\begin{aligned} h\left(\frac{x}{1}\right) = g\left(\frac{x}{1}\right) = g\left(\frac{y}{1}\cdot\frac{x}{y}\right) = g\left(\frac{y}{1}\right) g\left(\frac{x}{y}\right) = h\left(\frac{y}{1}\right) g\left(\frac{x}{y}\right) \\ \implies h\left(\frac{x}{y}\right) = h\left(\frac{y}{1}\right)^{-1} h\left(\frac{x}{1}\right) = g\left(\frac{x}{y}\right) \implies h = g. \end{aligned}$$

Hence, i is an epimorphism and, thus, a bimorphism. It is not an isomorphism because $\operatorname{Hom}_{\mathbf{Ring}}(\mathbb{Q},\mathbb{Z}) = \varnothing$ (suppose there is a ring homomorphism $k\colon \mathbb{Q} \to \mathbb{Z}$; then $2k(\frac{1}{2}) = 1$, which is a contradiction).

Lemma 1.2.19. *Let* $\mathbf{C}$ *be a category that has zero morphisms. If* $g\colon b \to c$ *is a monomorphism and for some* $f\colon a \to b$ *we have* $g \circ f = 0_{a,c}$*, then* $f = 0_{a,b}$*.*

Proof. Since $g \circ f = 0_{a,c} = g \circ 0_{a,b}$, we have $f = 0_{a,b}$. $\square$

Lemma 1.2.20. *Let* $\mathbf{C}$ *be a category that has a zero object* 0. *Then* $0_{a,a} = \mathrm{id}_a$ *if and only if* $a \cong 0$.

Proof. The verification is left to the reader. $\square$

Proposition 1.2.21. *Let* $\mathbf{C}$ *be a category and let* $f\colon a \to b$ *and* $g\colon b \to c$.

(a) If $g \circ f$ *is a monomorphism, then* f *is a monomorphism.*

(b) If $g \circ f$ *is an epimorphism, then* g *is an epimorphism.*

(c) If g *and* f *are monomorphisms, then* $g \circ f$ *is a monomorphism.*

(d) If g *and* f *are epimorphisms, then* $g \circ f$ *is an epimorphism.*

(e) If g *and* f *are bimorphisms, then* $g \circ f$ *is a bimorphism.*

Proof. (a) Suppose $g \circ f$ is a monomorphism and let $k\colon d \to a$ and $j\colon d \to a$ be arbitrary morphisms. We see that $f \circ k = f \circ j \implies g \circ f \circ k = g \circ f \circ j \implies k = j$ and, hence, f is monomorphism.

(b) Suppose $g \circ f$ is an epimorphism and let $k'\colon c \to d$, and $j'\colon c \to d$ be arbitrary morphisms. We see that $k' \circ g = j' \circ g \implies k' \circ g \circ f = j' \circ g \circ f \implies k' = j'$ and, hence, g is an epimorphism.

(c) Let f and g be monomorphisms and suppose there exist morphisms $h\colon d \to a$ and $h'\colon d \to a$ such that $(g \circ f) \circ h = (g \circ f) \circ h'$. Then, by associativity of composition, we have the following:

$$g \circ (f \circ h) = g \circ (f \circ h') \implies f \circ h = f \circ h' \implies h = h'$$

(d) The proof is similar to part (c).

(e) The statement is a direct consequence of parts (c) and (d). $\square$

Definition 1.2.22 (Split monomorphism/epimorphism)**.** A morphism $f\colon a \to b$ is a *split monomorphism* if there exists a morphism $g\colon b \to a$ such that $g \circ f = \mathrm{id}_a$. A morphism $f\colon a \to b$ is a *split epimorphism* if there exists a morphism $g\colon b \to a$ such that $f \circ g = \mathrm{id}_b$.

Lemma 1.2.23. *A morphism* $f\colon a \to b$ *is an isomorphism if and only if it is both a split monomorphism and a split epimorphism.*

Proof. The forward implication is clear. For the reverse implication, let $g\colon b \to a$ and $h\colon b \to a$ be the morphisms satisfying $g \circ f = \mathrm{id}_a$ and $f \circ h = \mathrm{id}_b$. Then, we have $g = g \circ \mathrm{id}_b = g \circ f \circ h = \mathrm{id}_a \circ h = h$. We conclude that f is an isomorphism with $g = h = f^{-1}$. $\square$

Lemma 1.2.24. *Split monomorphisms are monomorphisms and split epimorphisms are epimorphisms.*

Proof. Suppose $f\colon b \to c$ is a split monomorphism and that there exist morphisms $g\colon a \to b$ and $h\colon a \to b$ satisfying $f \circ g = f \circ h$. Then, by definition, there exists a morphism $j\colon c \to b$ such that $j \circ f = \mathrm{id}_b$, giving us the following:

$$g = \mathrm{id}_b \circ g = j \circ f \circ g = j \circ f \circ h = \mathrm{id}_b \circ h = h.$$

Hence, f is a monomorphism. The proof that split epimorphisms are epimorphisms is similar. $\square$

Definition 1.2.25 (Kernel)**.** Let $\mathbf{C}$ be a category that has zero morphisms. A *kernel* of a morphism $f\colon a \to b$, denoted $\ker(f)$, is a morphism $k\colon c \to a$ that satisfies the following conditions:

(a) $f \circ k = 0_{c,b}$.

(b) For any morphism $k'\colon d \to a$ such that $f \circ k' = 0_{d,b}$, there exists a unique morphism $g\colon d \to c$ such that $k \circ g = k'$.

This statement can be summarized by the following commutative diagram:

$$\begin{tikzcd}
c \arrow[r, "k"] \arrow[rr, bend left, "0_{c,b}"] & a \arrow[r, "f"] & b \\
d \arrow[u, dotted, "g"] \arrow[ur, "k'"] \arrow[urr, "0_{d,b}"'] & &
\end{tikzcd}$$

Lemma 1.2.26. *Let $\mathbf{C}$ be a category that has zero morphisms. If the kernel of a morphism exists, then the kernel is a monomorphism.*

Proof. Let $f\colon a \to b$ be a morphism that has a kernel $k\colon c \to a$. Suppose that there exist morphisms $g\colon d \to c$ and $h\colon d \to c$ such that $k \circ g = k \circ h$. Then, $f \circ k \circ g = 0_{c,b} \circ g = 0_{d,b}$ and hence, there exists a unique morphism $n\colon d \to c$ such that $k \circ g = k \circ n$. By uniqueness, we have $g = n = h$. $\square$

Lemma 1.2.27. *The kernel of a morphism is unique up to isomorphism.*

Proof. Let f be some morphism and suppose $\ker(f)$ and $\ker(f)'$ are kernels of f. By definition, there exists a unique morphism g such that $\ker(f) \circ g = \ker(f)'$ and a unique morphism g' such that $\ker(f)' \circ g' = \ker(f)$. We have

$$\ker(f) \circ (g \circ g') = \ker(f) \quad \text{and} \quad \ker(f)' \circ (g' \circ g) = \ker(f)'.$$

By Lemma 1.2.26, we note that $g \circ g'$ and $g' \circ g$ are identity morphism since a kernel is a monomorphism. In particular, g is an isomorphism and, thus, $\ker(f) \cong \ker(f)'$ since $\mathrm{id}_{\mathrm{cod}(\ker(f))} \circ \ker(f) \circ g = \ker(f)'$. $\square$

The kernel is an example of an object with a *universal property*. Objects with universal properties are always unique up to isomorphism. Henceforth, when an object has a universal property, we will take for granted that it is unique up to isomorphism, and that the justification would be similar to that of the previous proof.

Example 1.2.28. Consider the morphism $f\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ in **Ab** given by $f(z,w)=z$. We have $\ker(f)\cong k$ where $k\colon \mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ is given by $k(z)=(0,z)$. Uniqueness in the second condition of Definition 1.2.25 is important. The morphism $h\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ given by $h(z,w)=(0,z)$ satisfies the first condition of Definition 1.2.25 but fails the second because there is more than one morphism $g\colon \mathbb{Z}\times\{0\}\to\mathbb{Z}\times\mathbb{Z}$ satisfying $h\circ g=k$ (for instance, both $g_1(z,0)=(z,0)$ and $g_2(z,0)=(z,z)$ suffice).

Proposition 1.2.29. *The kernel of a homomorphism in* **Grp** *coincides with the traditional group theory kernel.*

Proof. Let $f\colon A\to B$ be a group homomorphism. Let $C=\{a\in A\mid f(a)=0_B\}$, which is the traditional kernel and hence a normal subgroup of A. We claim that $\ker(f)\cong i$ where $i\colon C\hookrightarrow A$ is the inclusion map. Obviously, $f\circ i=0_{C,B}$. Suppose there is some $k\colon D\to A$ such that $f\circ k=0_{D,B}$. Then, $k(D)\subseteq C$. Hence, $i\circ\tilde{k}=k$ where $\tilde{k}\colon D\to C$ is defined by $\tilde{k}(z)=k(z)$. The uniqueness of $\tilde{k}$ follows from the fact that i is injective and thus monic.

Henceforth, when working in **Grp** (or any other category with an algebraic notion of "kernel", such as **Ab**, $\mathbf{Vect}_{\mathbb{K}}$ and R-**Mod**), if $\ker(f)$ denotes an object, then we are implicitly referring to the object C defined above and assuming the inclusion map. $\square$

Example 1.2.30. If **C** has a zero object 0, then for any $a\in\mathbf{C}$, $\ker(\mathrm{id}_a)\cong 0_{0,a}$.

Definition 1.2.31 (Cokernel)**.** Let **C** be a category that has zero morphisms. The *cokernel* of a morphism $f\colon a\to b$ is a morphism $k\colon b\to c$, denoted $\operatorname{coker}(f)$, such that

(a) $k\circ f=0_{a,c}$

(b) For any morphism $k'\colon b\to d$ such that $k'\circ f=0_{a,d}$, there exists a unique morphism $g\colon c\to d$ such that $g\circ k=k'$.

This statement can be summarized by the following commutative diagram:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, bend left, "0_{a,c}"] \arrow[d, "0_{a,d}"'] & b \arrow[r, "k"] \arrow[dl, "k'"'] & c \arrow[dll, dotted, "g"] \\
d
\end{tikzcd}$$

1.3 (Co)products, Pullbacks & Pushouts

In this section, we present the general notion of products and coproducts that may exist in an arbitrary category. The reader is likely to be already familiar with some specific cases. Understanding these objects will be important in the study of Grothendieck groups, particularly the split Grothendieck group of an additive category.

Definition 1.3.1 (Binary product)**.** Let a and b be objects in a category. An object is a *binary product* of a and b, which we denote by $a\prod b$, if there exist morphisms $p_a\colon a\prod b \to a$ and $p_b\colon a\prod b \to b$ satisfying the property that for every object c and every pair of morphisms $f_a\colon c \to a$ and $f_b\colon c \to b$, there exists a unique morphism $f\colon c \to a\prod b$ such that the following diagram commutes:

$$\begin{tikzcd}
 & c \arrow[dl, "f_a"'] \arrow[d, dotted, "f"] \arrow[dr, "f_b"] & \\
a & a\prod b \arrow[l, "p_a"'] \arrow[r, "p_b"] & b
\end{tikzcd}$$

We will sometimes denote a binary product of a and b as a triple $(a\prod b, p_a, p_b)$ to emphasize the morphisms p_a and p_b.

Lemma 1.3.2. *The binary product is unique up to isomorphism.*

Proof. This follows directly from the universal property. □

Lemma 1.3.3. *In a category with zero morphisms, if $(a\prod b, p_a, p_b)$ is a product of a and b, then the morphisms p_a and p_b are split epimorphisms.*

Proof. There exists a unique morphism f that makes the following diagram commute:

$$\begin{tikzcd}
 & a \arrow[dl, "\mathrm{id}_a"'] \arrow[d, dotted, "f"] \arrow[dr, "0_{a,b}"] & \\
a & a\prod b \arrow[l, "p_a"'] \arrow[r, "p_b"] & b
\end{tikzcd}$$

Hence, $p_a \circ f = \mathrm{id}_a$ and so p_a is a split epimorphism. Similarly, one can show that p_b is a split epimorphism. □

Lemma 1.3.4. *If an object c is terminal, then $a \cong a\prod c$ for all objects a.*

Proof. Suppose c is terminal. Let a and b be an arbitrary objects and let $f_a\colon b \to a$ and $f_c\colon b \to c$ be arbitrary morphisms. For every object d, denote the unique morphism from $d \to c$ by c_d. Note that f_a makes the following diagram commute (since $c_a \circ f_a = c_b = f_c$):

$$\begin{tikzcd}
 & b \arrow[dl, "f_a"'] \arrow[d, dotted, "f_a"] \arrow[dr, "f_c"] & \\
a & a \arrow[l, "\mathrm{id}_a"'] \arrow[r, "c_a"] & c
\end{tikzcd}$$

Suppose some morphism $g\colon b \to a$ also makes this diagram commute. Then $g = \mathrm{id}_a \circ g = f_a$. Hence, a is a product of a and c and, thus, $a \cong a\prod c$. □

Remark 1.3.5. The converse of the preceding lemma need not be true. For example, take the full subcategory of **Set** where the only objects are countably infinite sets; there is no terminal object and yet $a \prod \mathbb{Z} \cong \mathbb{Z}$ for any object a.

Lemma 1.3.6. *Let* $\mathbf{C}$ *be a category and* $a, b, c \in \mathbf{C}$*. Then, we have the following:*

(a) Product is commutative up to isomorphism. That is, if $a \prod b$ *exists, then* $a \prod b \cong b \prod a$*.*

(b) Product is associative up to isomorphism. That is, if $a \prod b$*,* $b \prod c$*, and* $(a \prod b) \prod c$ *exist, then* $(a \prod b) \prod c \cong a \prod (b \prod c)$*.*

Proof. (a) This follows immediately from the symmetry in the definition of product.

(b) Fix products $(a \prod b, p_a, p_b)$, $(b \prod c, p'_b, p_c)$, and $((a \prod b) \prod c, p_{a \prod b}, p'_c)$. Since products are unique up to isomorphism, it suffices to show that $(a \prod b) \prod c$ is a product of a and $b \prod c$. For any $h_a \colon d \to a$ and $h_{b \prod c} \colon d \to b \prod c$ there are unique morphisms $g \colon (a \prod b) \prod c \to b \prod c$, $f \colon d \to a \prod b$ and $r \colon d \to (a \prod b) \prod c$ such that the following diagrams commute:

$$\begin{tikzcd}
& (a \prod b) \prod c \arrow[dl, "p_b \circ p_{a \prod b}"'] \arrow[d, dotted, "g"] \arrow[dr, "p'_c"] & \\
b & b \prod c \arrow[l, "p'_b"'] \arrow[r, "p_c"] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
& d \arrow[dl, "h_a"'] \arrow[d, dotted, "f"] \arrow[dr, "p'_b \circ h_{b \prod c}"] & \\
a & a \prod b \arrow[l, "p_a"'] \arrow[r, "p_b"] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
& d \arrow[dl, "f"'] \arrow[d, dotted, "r"] \arrow[dr, "p_c \circ h_{b \prod c}"] & \\
a \prod b & (a \prod b) \prod c \arrow[l, "p_{a \prod b}"'] \arrow[r, "p'_c"] & c
\end{tikzcd}$$

We note that

$$p'_b \circ (g \circ r) = p_b \circ p_{a \prod b} \circ r = p_b \circ f = p'_b \circ h_{b \prod c}$$
and
$$p_c \circ (g \circ r) = p'_c \circ r = p_c \circ h_{b \prod c}.$$

Hence, $h_{b \prod c} = g \circ r$ and the following diagram commutes:

$$\begin{tikzcd}
& d \arrow[dl, "h_a"'] \arrow[d, dotted, "r"] \arrow[dr, "h_{b \prod c}"] & \\
a & (a \prod b) \prod c \arrow[l, "p_a \circ p_{a \prod b}"'] \arrow[r, "g"] & b \prod c
\end{tikzcd}$$

Furthermore, if some $r' \colon d \to (a \prod b) \prod c$ satisfies $(p_a \circ p_{a \prod b}) \circ r' = h_a$ and $g \circ r' = h_{b \prod c}$. Then,

$$p_a \circ (p_{a \prod b} \circ r') = h_a = p_a \circ f \quad \text{and} \quad p_b \circ (p_{a \prod b} \circ r') = p'_b \circ g \circ r' = p'_b \circ h_{b \prod c} = p_b \circ f.$$

Hence, $f = p_{a \prod b} \circ r'$. Then, since we also have $p'_c \circ r' = p_c \circ g \circ r' = p_c \circ h_{b \prod c}$, the uniqueness in the product diagram for $(a \prod b) \prod c$ yields $r = r'$. Therefore, $(a \prod b) \prod c \cong a \prod (b \prod c)$. $\square$

Definition 1.3.7 (Binary coproduct)**.** Let a and b be objects in a category. An object is the *binary coproduct* of a and b, and is denoted $a \coprod b$, if there exist morphisms $i_a \colon a \to a \coprod b$ and $i_b \colon b \to a \coprod b$ satisfying the property that for every object c and every pair of morphisms $f_a \colon a \to c$ and $f_b \colon b \to c$, there exists a unique morphism $f \colon a \coprod b \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
 & c & \\
a \arrow[ur, "f_a"] \arrow[r, "i_a"] & a \coprod b \arrow[u, dotted, "f"] & b \arrow[ul, "f_b"'] \arrow[l, "i_b"']
\end{tikzcd}$$

As with the product, we will sometimes denote a binary coproduct as a triple $(a \coprod b, i_a, i_b)$ to emphasize the morphisms i_a and i_b.

Remark 1.3.8. The dual notions of the preceding results for the product hold for the coproduct. Coproduct is also unique up to isomorphism, as well as associative and commutative up to isomorphism. In a category with zero morphisms, the morphisms i_a and i_b in Definition 1.3.7 are split monomorphisms. If c is an initial object, then $a \coprod c \cong a$ for all objects a.

Example 1.3.9. Let $A, B \in \mathbf{Set}$. Then the product of A and B is the cartesian product $A \times B$ and the coproduct is the disjoint union $A \dot\cup B = \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$. Define $\pi_1 \colon A \times B \to A$ and $\pi_2 \colon A \times B$ to be the usual projection maps. Define $\iota_1 \colon A \to A \dot\cup B$ and $\iota_2 \colon B \to A \dot\cup B$ to be the usual inclusion maps. For any set X and any functions $f_a \colon X \to A$ and $f_b \colon X \to B$, define:

$$\begin{aligned} f \colon X &\to A \times B, \\ x &\mapsto (f_a(x), f_b(x)). \end{aligned}$$

For any functions $g_a \colon A \to X$ and $g_b \colon B \to X$, define $g \colon A \dot\cup B \to X$ by:

$$g(x) = \begin{cases} g_a(x) & \text{if } x \text{ is of the form } (a, 0),\ a \in A, \\ g_b(x) & \text{if } x \text{ is of the form } (b, 1),\ b \in B. \end{cases}$$

This gives us the following commutative diagrams:

$$\begin{tikzcd}
 & X \arrow[dl, "f_a"'] \arrow[d, dotted, "f"] \arrow[dr, "f_b"] & \\
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X & \\
A \arrow[ur, "g_a"] \arrow[r, "\iota_1"] & A \dot\cup B \arrow[u, dotted, "g"] & B \arrow[ul, "g_b"'] \arrow[l, "\iota_2"']
\end{tikzcd}$$

We leave the verification that f and g are unique to the reader.

Definition 1.3.10 (Pullback square, pullback)**.** Let $f \colon X \to Z$ and $g \colon Y \to Z$ be morphisms with a common codomain. A *pullback square* is a commutative square consisting of an object P and morphisms $p_1 \colon P \to X$ and $p_2 \colon P \to Y$:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

such that for any commutative square of the form

$$\begin{tikzcd} Q \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

there exists a unique morphism $k\colon Q \to P$ which makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[dr, dotted, "k"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

We call $X \overset{p_1}{\leftarrow} P \overset{p_2}{\to} Y$ a *pullback* of $X \overset{f}{\to} Z \overset{g}{\leftarrow} Y$ wherein p_1 is the *pullback of g along f* and p_2 is the *pullback of f along g*.

Definition 1.3.11 (Pushout square, pushout)**.** Let $f\colon Z \to X$ and $g\colon Z \to Y$ be morphisms with a common domain. A *pushout square* is a commutative square consisting of an object P and morphisms $i_1\colon X \to P$ and $i_2\colon Y \to P$:

$$\begin{tikzcd} P & Y \arrow[l, "i_2"'] \\ X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

such that for any commutative square of the form

$$\begin{tikzcd} Q & Y \arrow[l, "j_2"'] \\ X \arrow[u, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

there exists a unique morphism $u\colon P \to Q$ which makes the following diagram commute:

$$\begin{tikzcd} Q & & \\ & P \arrow[ul, dotted, "u"] & Y \arrow[l, "i_2"'] \arrow[ull, bend right, "j_2"'] \\ & X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

We call $X \overset{i_1}{\to} P \overset{i_2}{\leftarrow} Y$ a *pushout* of $X \overset{f}{\leftarrow} Z \overset{g}{\to} Y$ wherein i_1 is the *pushout of g along f* and i_2 is the *pushout of f along g*.

Lemma 1.3.12. *Pullbacks of kernels are kernels.*

Proof. We use the objects and morphisms as defined in Definition 1.3.10. Let $A \in \mathbf{C}$ and $h\colon Z \to A$ be a morphism. We will show that if $f \cong \ker h$ (given that p_2 is the pullback of f along g and p_1 is the pullback of g along f), then $p_2 \cong \ker(h \circ g)$.

Let K be an object and $k_2\colon K \to Y$ be a morphism such that $h \circ g \circ k_2 = 0_{K,A}$. Then, since $f \cong \ker h$, there is a unique morphism $k_1\colon K \to X$ such that $k_2 \circ g = k_1 \circ f$. By the pullback property, there is a unique morphism k which makes the following diagram commute:

$$\begin{tikzcd}
K \arrow[dr, dotted, "k"] \arrow[drr, bend left, "k_2"] \arrow[dddrrr, bend left, "0_{K,A}"] \arrow[ddr, dotted, "k_1"'] & & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] & \\
& X \arrow[r, "f"'] \arrow[drr, bend right, "0_{X,A}"'] & Z \arrow[dr, "h"] & \\
& & & A
\end{tikzcd}$$

Since the composition of a zero morphism with any morphism is again a zero morphism, we have that $h \circ f \circ p_1 = 0_{X,A} \circ p_1 = 0_{P,A}$ and hence $h \circ g \circ p_2 = 0_{P,A}$. Since K and k_2 were arbitrary, we see from the above diagram that $p_2 \cong \ker(h \circ g)$, as desired. $\square$

Lemma 1.3.13. *Pushouts of cokernels are cokernels.*

Proof. The proof, which is dual to that of the previous lemma, is left to the reader. $\square$

1.4 Subobjects & Simple Objects

The reader is already familiar with the structure of many subobjects for specific categories including subsets in **Set**, subgroups in **Grp** and subrings in **Ring**. In this section, we will use morphisms to define the notion of a subobject of an object in an arbitrary category. The notion of subobjects will be important for when we present the Jordan-Hölder Theorem.

Definition 1.4.1 (Subobject)**.** Let $\mathbf{C}$ be a category and let $a \in \mathbf{C}$. We define the following preorder on $\{k \in \operatorname{Mor} \mathbf{C} \mid \operatorname{cod}(k) = a\}$:

If $f\colon b \to a$ and $g\colon c \to a$ are monomorphisms, then

$$f \leq g \iff \text{there exists a monomorphism } h\colon b \to c \text{ such that } f = g \circ h.$$

If $f \leq g$ and $g \leq f$, we write $f \equiv g$. An equivalence class of the relation $\equiv$ is called a *subobject* of a.

Remark 1.4.2. For some special categories like **Set** and **Grp**, we identify each subobject with a distinct object (usually the domain of some morphism in the equivalence class). This cannot be done in general; for example, the object b of the category $\downdownarrows$ given in Example 1.1.9 has three subobjects, but there are only two objects in this category and so each subobject cannot be identified with a distinct object.

The reader may verify that the preorder $\leq$ from Definition 1.4.1 induces a partial order on the subobjects of an object. We interpret this partial order as a category in the following way:

Definition 1.4.3 ($\mathbf{Sub}(a)$)**.** Let $\mathbf{C}$ be a category and let $a \in \mathbf{C}$. The category $\mathbf{Sub}(a)$ is given by the following data:

(a) $\operatorname{Ob}\mathbf{Sub}(\mathbf{C})$ consists of the subobjects of a in $\mathbf{C}$.

(b)
$$\operatorname{Hom}_{\mathbf{Sub}(a)}(X, Y) = \begin{cases} \{f_{X,Y}\} & \text{if } X \leq Y \\ \varnothing & \text{otherwise} \end{cases}$$

Every object has an identity morphism since partial orders are reflexive. Since each hom-class has at most one morphism, composition is completely determined by the domain and codomain of the morphisms. Furthermore, the transitivity of the partial order ensures that composition is well defined.

Example 1.4.4. In **Grp** and **Ab**, the notion of subobjects coincides with the notion of subgroups. For each subobject of a group G, the image of a representative morphism corresponds to a subgroup of G. Moreover, each subgroup corresponds to the subobject represented by its inclusion map. The partial order given by $\leq$ corresponds to $\subseteq$, and the category $\mathbf{Sub}(G)$ corresponds to the Hasse diagram of this partial order.

Definition 1.4.5 (Power class, power set)**.** Let $\mathbf{C}$ be a category and $A \in \mathbf{C}$. The partially ordered class of subobjects of A is called the *power class* of A. If $\mathbf{C}$ is locally small, it follows that power classes are sets, in which case they are called *power sets*. We will denote the power set of A by $\mathcal{P}(A)$ when this will not cause confusion with the set-theoretic definition of power sets.

Definition 1.4.6 (Minimal/maximal subobject)**.** Let $\mathbf{C}$ be a locally small category, $A \in \mathbf{C}$, and let U be a subset of $\mathcal{P}(A)$. An object $B \in U$ is *minimal* in U if $C \in U$ and $C \subseteq B$ implies $C = B$. Similarly, B is *maximal* in U if $C \in U$ and $B \subseteq C$ implies $B = C$. When we simply say that B is a minimal (respectively, maximal) subobject of A, we mean that it is minimal (respectively, maximal) in $\mathcal{P}(A) \setminus \{A\}$.

Definition 1.4.7 (Intersections and unions of subobjects)**.** Let $\mathbf{C}$ be a category with $a \in \mathbf{C}$. Let X_i be a collection of subobjects of a indexed by elements $i \in I$ for some non-empty set I. A *greatest lower bound* of the X_i is a subobject Y of a such that

(a) $Y \leq X_i$ for all $i \in I$,

(b) If for some subobject Z of a we have $Z \leq X_i$ for all $i \in I$, then $Z \leq Y$.

This is also called a *meet* or *intersection* of the X_i and is denoted $\cap X_i$ or $X_1 \cap \cdots \cap X_n$ if $I = \{1, \ldots, n\}$. A *least upper bound* of the X_i is a subobject Y of a such that

(a) $X_i \leq Y$ for all $i \in I$,

(b) If for some subobject Z of a we have $X_i \leq Z$ for all $i \in I$, then $Y \leq Z$.

This is also called a *join* or *union* of the X_i and is denoted $\cup X_i$ or $X_1 \cap \cdots \cap X_n$ if $I = \{1, \ldots, n\}$.

Remark 1.4.8. Let $\mathbf{C}$ be a category and $a \in \mathbf{C}$. The meet of two subobjects of a is a product of the two as objects in $\mathbf{Sub}(a)$. The join of two subobjects of a is a coproduct of the two as objects in $\mathbf{Sub}(a)$.

Definition 1.4.9 (Simple object)**.** A *simple object* is an object that has precisely two subobjects – one having the zero morphism as a representative and one having the identity as a representative. Equivalently, in a category with a zero object, an object A is simple if and only if the zero object is a maximal subobject of A.

Examples 1.4.10. (a) In **Grp** and **Ab**, maximal objects are precisely maximal subgroups and simple objects are precisely simple groups.

(b) In R-**Mod**, maximal objects are precisely maximal submodules and simple objects are precisely simple modules.

Remark 1.4.11. It is important to note that intersections and unions of subobjects are *not* dual to each other. Their respective dual notions are cointersections and counions, but we will not discuss them in this paper. The dual notion to subobjects are *quotient objects*, which will be discussed in further detail in Section 2.5.

1.5 Functors & Natural Transformations

As we have already seen, it is the morphisms rather than the objects that play the lead role of interest in category theory. We now take a bird's eye view and consider categories themselves as "objects" with certain structure and discuss various morphisms between categories.

Definition 1.5.1 (Functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a function which assigns to each object $c \in \mathbf{C}$ an object $F(c) \in \mathbf{D}$, to each morphism $c \xrightarrow{f} c' \in \mathbf{C}$ a morphism $F(c) \xrightarrow{F(f)} F(c') \in \mathbf{D}$, and satisfies the properties:

(a) Preservation of identity: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for every object $c \in \mathbf{C}$,

(b) Preservation of composition: if g, f are composable, then $F(g \circ f) = F(g) \circ F(f)$.

We will write $F : \mathbf{C} \to \mathbf{D}$ or $\mathbf{C} \xrightarrow{F} \mathbf{D}$ to mean that F is a functor from $\mathbf{C}$ to $\mathbf{D}$.

Remark 1.5.2. Strictly speaking, a functor $F : \mathbf{C} \to \mathbf{D}$ is a family of functions which consists of a function F_{Ob} from the objects of $\mathbf{C}$ to the objects of $\mathbf{D}$ together with a function $F_{c,c'}$ from $\mathrm{Hom}(c, c')$ to $\mathrm{Hom}(F(c), F(c'))$ for each pair (c, c') of objects in $\mathbf{C}$. Furthermore, since functors preserve identity morphisms and since there is an obvious bijective correspondence between objects and identity morphisms, the action of a functor on objects is completely determined by its action on morphisms.

Example 1.5.3. Just like morphisms between objects in a category, we may compose functors when their domains and codomains agree. That is, for three categories $\mathbf{C}, \mathbf{D}$, and $\mathbf{E}$ with functors F and G of the form

$$\mathbf{C} \xrightarrow{G} \mathbf{D} \xrightarrow{F} \mathbf{E},$$

we define the *composite functor* to be the following functor:

$$\begin{aligned} F \circ G \colon \mathbf{C} &\to \mathbf{E}, \\ c &\mapsto F(G(c)) && \text{for all } c \in \mathbf{C}, \\ f &\mapsto F(G(f)) && \text{for all } f \in \mathrm{Mor}\,\mathbf{C}. \end{aligned}$$

Since this composition is associative and each category $\mathbf{C}$ has an identity functor $\mathrm{id}_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{C}$ (which simply maps all objects and morphisms of $\mathbf{C}$ to themselves), we can now define **Cat**, the *category of all small categories*, which has all small categories as objects and functors as morphisms.

Remark 1.5.4. We define **Cat** to be the category of all small categories rather than the category of all categories as doing so would cause a contradiction analogous to Russell's Paradox with the definition of "category" that we have developed. It is *possible* to axiomatize the category of all categories by adopting different foundational principles (in which category theory is considered more "fundamental" than set theory; that is, set theory is developed in terms of category theory). A detailed discussion is beyond the scope of this paper, but interested readers are encouraged to consult [13].

Example 1.5.5. The *power set functor* $\mathcal{P} \colon \mathbf{Set} \to \mathbf{Set}$ is the functor which sends every set X to its power set $\mathcal{P}(X)$ (the elements of which are the subsets of X), and which sends each function $f \colon X \to Y$ to the function $\mathcal{P}(f) \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ which maps each subset $Z \subseteq X$ to its image under f, $f(Z) \subseteq Y$.

Example 1.5.6. Let R, S be commutative rings. Recall that $\mathrm{GL}_n(R)$, the *general linear group* of degree n over R, is the multiplicative group of all invertible $n \times n$ matrices with entries in R. Now let $\varphi \colon R \to S$ be an arbitrary ring homomorphism. Define a map:

$$\mathrm{GL}_n\varphi \colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(S),$$
$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{pmatrix}.$$

A routine verification reveals that $\mathrm{GL}_n\varphi$ is a group homomorphism. This data, which arises naturally in algebra, defines an infinite family of functors; for each $n \in \mathbb{N}^+$ we have the functor $\mathrm{GL}_nF\colon \mathbf{CRing} \to \mathbf{Grp}$ which sends each commutative ring R to $\mathrm{GL}_n(R)$ and which sends each ring homomorphism $\varphi\colon R \to S$ to the group homomorphism $\mathrm{GL}_n\varphi\colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$.

Examples 1.5.7. A functor is *forgetful* if it simply causes us to lose some information (a more precise definition of this term will be given in Section 3.2). For instance:

(a) Let $U\colon \mathbf{Grp} \to \mathbf{Set}$ be the functor which sends each group to its underlying set and which sends each group homomorphism to the corresponding function between sets. Thus, U is a forgetful functor whence we have "forgotten" the group operation.

(b) Let $V\colon \mathbf{Ring} \to \mathbf{Grp}$ be the functor which sends each ring to its additive group and which sends each ring homomorphism to the corresponding group homomorphism. Thus, V is a forgetful functor whence we have "forgotten" the multiplication operation.

(c) The composite functor $U \circ V\colon \mathbf{Ring} \to \mathbf{Set}$ sends each ring to its underlying set and sends each ring homomorphism to the corresponding function between sets. Thus, $U \circ V$ "forgets" both of the ring operations.

Definition 1.5.8 (Isomorphism of categories)**.** Two categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* if there exist functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G = \mathrm{id}_{\mathbf{D}}$. When this is the case, we say that G is the *two-sided inverse* of F and write $G = F^{-1}$.

Lemma 1.5.9. *Let* $\mathbf{C}$ *be some category. A skeleton subcategory of* $\mathbf{C}$ *is unique up to isomorphism.*

Proof. Suppose $\underline{\mathbf{C}}$ and $\underline{\mathbf{C}}'$ are two skeleton subcategories of $\mathbf{C}$. Let $F\colon\ \mathrm{Ob}\,\underline{\mathbf{C}} \to \mathrm{Ob}\,\underline{\mathbf{C}}'$ assign to each object of $\underline{\mathbf{C}}$ the unique object in $\underline{\mathbf{C}}'$ that is isomorphic to it in $\mathbf{C}$. Clearly, every object in $\underline{\mathbf{C}}'$ can be written in the form $F(a)$ for some $a \in \underline{\mathbf{C}}$. For every object $a \in \mathrm{Ob}\,\underline{\mathbf{C}}$, fix an isomorphism $f_a\colon a \to F(a)$. We leave it to the reader to verify that the maps

$$\begin{aligned} F\colon \underline{\mathbf{C}} &\to \underline{\mathbf{C}}' \\ a &\mapsto F(a) \\ (f\colon a \to b) &\mapsto f_b \circ f \circ f_a^{-1} \end{aligned} \qquad \text{and} \qquad \begin{aligned} G\colon \underline{\mathbf{C}}' &\to \underline{\mathbf{C}} \\ F(a) &\mapsto a \\ (f'\colon F(a) \to F(b)) &\mapsto f_b^{-1} \circ f' \circ f_a \end{aligned}$$

quantified over the objects and morphisms of their respective categories, are mutually inverse functors (i.e. they satisfy Definition 1.5.1 and Definition 1.5.8). □

Definition 1.5.10 (Faithful/full functor)**.** As we have noted in Remark 1.5.2, the data of a functor $F\colon \mathbf{C} \to \mathbf{D}$ includes a function $F_{c,c'}\colon\ \mathrm{Hom}(c, c') \to \mathrm{Hom}(F(c), F(c'))$ for each pair (c, c') of objects in $\mathbf{C}$. If all of these functions on hom-classes are injective, we say that F is *faithful* . If they are all surjective, we say that F is *full*. A functor which is both faithful and full is *fully faithful*.

Remark 1.5.11. A faithful functor need not be injective on either objects or morphisms and a full functor need not be surjective on either objects or morphisms.

Examples 1.5.12. (a) The forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ is faithful but not full (for instance, there are functions from $\mathbb{R}$ to $\mathbb{R}$ which are not group homomorphisms and thus not in the image of $U_{\mathbb{R},\mathbb{R}}$).

(b) Let $\mathbf{C}$ be some category and let $\underline{\mathbf{C}}$ be a skeleton of $\mathbf{C}$. The natural inclusion functor $I\colon \underline{\mathbf{C}} \hookrightarrow \mathbf{C}$ is fully faithful. If two distinct objects are isomorphic in $\mathbf{C}$, then this functor is surjective on neither objects nor morphisms.

(c) Recall the category $\mathbf{1}$ from Example 1.1.9. Let $F\colon \mathbf{Set} \to \mathbf{1}$ be the functor which sends every set to the unique object a of $\mathbf{1}$ and every function to id_a. Then, F is full but not faithful.

(d) Recall the category $\downdownarrows$ from Example 1.1.9. Let $G\colon \downdownarrows \to \mathbf{Set}$ be the functor which sends a to $\mathbb{Z}$, b to $\mathbb{Q}$, and both f and g to the inclusion map. Then, G is neither full nor faithful.

Definition 1.5.13 (Covariant/contravariant functor)**.** The functors we have discussed thus far are *covariant functors* in that they preserve the direction of morphisms. We will continue to simply say "functor" to mean a covariant functor. A *contravariant functor* $F\colon \mathbf{C} \to \mathbf{D}$, on the other hand, reverses morphisms in that it assigns to each morphism $f\colon A \to B \in \operatorname{Mor}\mathbf{C}$ a morphism $F(f)\colon F(B) \to F(A) \in \operatorname{Mor}\mathbf{D}$ and composable morphisms $f, g \in \operatorname{Mor}\mathbf{C}$ satisfy $F(g \circ f) = F(f) \circ F(g)$. A contravariant functor otherwise satisfies the same properties as a covariant functor.

Remark 1.5.14. A contravariant functor $F\colon \mathbf{C} \to \mathbf{D}$ is equivalent to a covariant functor $F'\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.

Example 1.5.15. Fix a field $\mathbb{K}$ and consider $\mathbf{FinVect}_{\mathbb{K}}$. Recall that every vector space V has a *dual space* $V^* = \mathrm{Hom}(V, \mathbb{K})$. Define the *dual functor* $D\colon \mathbf{FinVect}_{\mathbb{K}} \to \mathbf{FinVect}_{\mathbb{K}}$ to be the functor which sends each $\mathbb{K}$-vector space V to its dual V^* and each linear transformation $T\colon V \to W$ to its transpose $T^*\colon W^* \to V^*$ defined by $T^*(g) = g \circ T$ for all $g \in W^*$. Hence, D is a contravariant functor.

Definition 1.5.16 (Bifunctor)**.** Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ be categories. A *bifunctor* $B\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is a functor from the product category $\mathbf{C} \times \mathbf{D}$ to $\mathbf{E}$. When unspecified, we assume a bifunctor is covariant in both arguments. We may consider a bifunctor which is contravariant in one or both arguments by considering $\mathbf{C}^{\mathrm{op}}$ or $\mathbf{D}^{\mathrm{op}}$ in place of $\mathbf{C}$ or $\mathbf{D}$, respectively, in the product category.

Example 1.5.17. Fix a field $\mathbb{K}$ and consider $\mathbf{Vect}_{\mathbb{K}}$ with the tensor product $\otimes$ defined in the usual way. This defines a bifunctor $\otimes\colon \mathbf{Vect}_{\mathbb{K}} \times \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Vect}_{\mathbb{K}}$, covariant in both arguments, which sends each pair of $\mathbb{K}$-vector spaces (U, V) to $U \otimes V$ and each pair of linear maps $(T\colon V \to V', S\colon W \to W')$ to $T \otimes S\colon V \otimes W \to V' \otimes W'$.

Definition 1.5.18 (Hom-functor)**.** Let $\mathbf{C}$ be a locally small category with $A, B \in \mathbf{C}$. Throughout this definition, all hom-classes are in $\mathbf{C}$. We define the *covariant hom-functor* to be the following functor:

$$\begin{aligned} \operatorname{Hom}(A, -)\colon \mathbf{C} &\to \mathbf{Set}, \\ X &\mapsto \operatorname{Hom}(A, X) && \text{for all } X \in \mathbf{C}, \\ f &\mapsto \operatorname{Hom}(A, f) && \text{for all } f \in \operatorname{Mor}\mathbf{C}, \end{aligned}$$

where for every morphism f we have $\operatorname{Hom}(A, f)(g) = f \circ g$ for all $g \in \operatorname{Hom}(A, \operatorname{dom}(f))$. Similarly, we define the *contravariant hom-functor* to be the following functor:

$$\begin{aligned} \operatorname{Hom}(-, B)\colon \mathbf{C} &\to \mathbf{Set}, \\ X &\mapsto \operatorname{Hom}(X, B) && \text{for all } X \in \mathbf{C}, \\ h &\mapsto \operatorname{Hom}(h, B) && \text{for all } h \in \operatorname{Mor}\mathbf{C}, \end{aligned}$$

where for every morphism h we have $\operatorname{Hom}(h, B)(g) = g \circ h$ for all $g \in \operatorname{Hom}(\operatorname{cod}(h), B)$.

Definition 1.5.19 (Natural transformation, natural isomorphism)**.** Let $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* $\tau\colon F \to G$ (or $F \xrightarrow{\tau} G$) is a function:

$$\begin{aligned} \operatorname{Ob}\mathbf{C} &\to \operatorname{Mor}\mathbf{D}, \\ a &\mapsto (F(a) \xrightarrow{\tau_a} G(a)), \end{aligned}$$

such that for every morphism $a \xrightarrow{f} a'$, the following diagram commutes:

$$\begin{tikzcd} F(a) \arrow[r, "\tau_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(a') \arrow[r, "\tau_a'"'] & G(a') \end{tikzcd}$$

Natural transformations are also called *morphisms of functors* . A natural transformation $\tau\colon F \to G$ is a *natural isomorphism* if for every object $a \in \mathbf{C}$, the function $\tau_a\colon F(a) \to G(a)$ is invertible. When this is the case, the inverses $(\tau_a)^{-1}$ are the component functions of the natural isomorphism $\tau^{-1}\colon G \to F$ and we write $\tau\colon F \cong G$.

Example 1.5.20. Consider the forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ and define a new functor $S\colon \mathbf{Grp} \to \mathbf{Set}$:

$$\begin{aligned} \operatorname{Ob}\mathbf{Grp} &\to \operatorname{Ob}\mathbf{Set}, & \operatorname{Mor}\mathbf{Grp} &\to \operatorname{Mor}\mathbf{Set}, \\ G &\mapsto G \times G, & (G \xrightarrow{\phi} H) &\mapsto (G \times G \xrightarrow{\phi\times\phi} H \times H). \end{aligned}$$

Now for any group $(G, \cdot)$, define the function (where the domain and codomain are objects of $\mathbf{Set}$):

$$\begin{aligned} \tau_G\colon G \times G &\to G, \\ (x, y) &\mapsto x \cdot y. \end{aligned}$$

Now fix an arbitrary group homomorphism $\phi\colon G \to H$ and consider the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\tau_G"] \arrow[d, "\phi\times\phi"'] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "\tau_H"'] & H \end{tikzcd}$$

This simply translates into the statement that for any $x, y \in G$, $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ – a true statement by the definition of a group homomorphism. Hence, this diagram commutes and so the family $\tau = \{\tau_G \mid G \in \mathbf{Grp}\}$ defines a natural transformation from S to U. Note that in the above diagram, all the "action" occurs in the category of sets – this is fine, since we have defined τ_G on sets rather than on groups. The forgetful functor may have forgotten the group operation, but *we* haven't!

Example 1.5.21. Fix a field $\mathbb{K}$ and consider $\mathbf{FinVect}_{\mathbb{K}}$. Recall that any vector space V over $\mathbb{K}$ has a *double dual*, $V^{**} = \mathrm{Hom}(\mathrm{Hom}(V, \mathbb{K}), \mathbb{K})$. It is known that when V is finite-dimensional, $V \cong V^{**}$, and that the linear map

$$\begin{aligned} \tau_V \colon V &\to V^{**}, \\ x &\mapsto (f \mapsto f(x)) \text{ for any } f \in V^*, \end{aligned}$$

is an isomorphism (we will accept this fact without proof – details can be found in [9, Theorem 2.26]). We thus have a canonical one-to-one identification between elements of V and V^{**} and so for any linear map $T\colon V \to W$, we have the map $T^{**}\colon V^{**} \to W^{**}$ (we may obtain T^{**} by applying the transpose twice, but to see the naturality of this identification more explicitly: if $\dim_{\mathbb{K}}(V) = n$ and we choose an arbitrary basis $\{v_1, \ldots, v_n\}$ for V, then the matrix of T will be the same as the matrix for T^{**} in the basis $\{\tau_V(v_1), \ldots, \tau_V(v_n)\}$ of V^{**}). Now consider the identity functor $\mathrm{id}_{\mathbf{FinVect}_{\mathbb{K}}}$ and the *double dual functor* $DD\colon \mathbf{FinVect}_{\mathbb{K}} \to \mathbf{FinVect}_{\mathbb{K}}$ which sends each vector space to its double dual and each linear map $T\colon V \to W$ to $T^{**}\colon V^{**} \to W^{**}$. We have the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\tau_V"] \arrow[d, "T"'] & V^{**} \arrow[d, "T^{**}"] \\ W \arrow[r, "\tau_W"'] & W^{**} \end{tikzcd}$$

Hence, the family $\tau = \{\tau_V \mid V \in \mathbf{FinVect}_{\mathbb{K}}\}$ defines a natural isomorphism from $\mathrm{id}_{\mathbf{FinVect}_{\mathbb{K}}}$ to DD.

Remark 1.5.22. Even though it is also true in the finite-dimensional case that $V \cong V^*$, it does not make sense to ask whether the dual functor D defined in Example 1.5.15 is naturally isomorphic to $\mathrm{id}_{\mathbf{FinVect}_{\mathbb{K}}}$ since, considered as covariant functors, D is a functor from ${\mathbf{FinVect}_{\mathbb{K}}}^{\mathrm{op}}$ to $\mathbf{FinVect}_{\mathbb{K}}$ whilst $\mathrm{id}_{\mathbf{FinVect}_{\mathbb{K}}}$ is a functor from $\mathbf{FinVect}_{\mathbb{K}}$ to $\mathbf{FinVect}_{\mathbb{K}}$. An isomorphism between a finite-dimensional vector space and its dual space, unlike the case of its double dual, is dependent on a choice of basis.

Definition 1.5.23 (Equivalence, Duality)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Covariant functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ are said to be *equivalences of categories* if natural isomorphisms

$$G \circ F \xrightarrow{\tau} \mathrm{id}_{\mathbf{C}} \quad \text{and} \quad F \circ G \xrightarrow{\eta} \mathrm{id}_{\mathbf{D}}$$

exist. If this is the case, then $\mathbf{C}$ and $\mathbf{D}$ are said to be *equivalent*. If the same holds except that F and G are contravariant functors, then we say $\mathbf{C}$ and $\mathbf{D}$ are *dually equivalent* and F and G are a *duality of categories*.

Definition 1.5.24 (Essentially surjective)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *essentially surjective* if for every $d \in \mathbf{D}$ there is some $c \in \mathbf{C}$ such that $F(c) \cong d$.

Lemma 1.5.25. *Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if F is a fully faithful essentially surjective functor.*

Proof. See [15, p. 93] for a proof. □

Definition 1.5.26 (Isomorphism reflecting)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *isomorphism reflecting* if for every $c, c' \in \operatorname{Ob}\mathbf{C}$ we have $F(c) \cong F(c') \implies c \cong c'$.

Chapter 2

The Grothendieck Group

We now reach the main chapter of the paper, where we will investigate the structure of Grothendieck groups of various types of categories. We will begin by discussing binary operations on categories, much like binary operations on a set, and see how this data leads us to our first examples of Grothendieck groups. We then move on to special types of categories that generalize structures occurring in various areas of mathematics, and see how their constructions also give rise to different Grothendieck groups.

2.1 Binary Operations on a Category

We say that a category is *nonempty* if its class of objects is nonempty. It follows that the class of morphisms of a nonempty category is also nonempty since each object has an identity morphism.

Definition 2.1.1 (Binary operation on a category)**.** Let $\mathbf{C}$ be a nonempty category. A *binary operation on* $\mathbf{C}$ is a bifunctor of the following form:

$$\begin{aligned} *\colon \mathbf{C} \times \mathbf{C} &\to \mathbf{C}, \\ (a,b) &\mapsto a * b \quad \text{for all } a, b \in \mathbf{C} \\ (f,g) &\mapsto f * g \quad \text{for all } f, g \in \operatorname{Mor} \mathbf{C}. \end{aligned} \tag{2.1.1}$$

Note that we will sometimes will write $*(a,b)$ and $*(f,g)$ instead of $a*b$ and $f*g$ to emphasize the action on the product category.

From now on, we will assume that all categories discussed are nonempty.

Lemma 2.1.2. *Let $*$ be a binary operation on a category $\mathbf{C}$. Then, we have the following:*

(a) For any objects $a, b \in \mathbf{C}$, we have $\mathrm{id}_a * \mathrm{id}_b = \mathrm{id}_{a*b}$.

(b) If for some morphisms g, f, g', f' we have that (g,f) and (g',f') are composable pairs, then $(g \circ f) * (g' \circ f') = (g * g') \circ (f * f')$.

Proof. (a) Functors preserve identity morphisms and $\mathrm{id}_{(a,b)} = (\mathrm{id}_a, \mathrm{id}_b)$ in the product category, so we have

$$\mathrm{id}_a * \mathrm{id}_b = *(\mathrm{id}_a, \mathrm{id}_b) = \mathrm{id}_{*(a,b)} = \mathrm{id}_{a*b}\,.$$

(b) By the preservation of composition, we have

$$(g \circ f) * (g' \circ f') = *(g \circ f, g' \circ f') = *((g, g') \circ (f, f')) = (g * g') \circ (f * f'). \qquad \square$$

Recall that morphism f and g are isomorphic if and only if $f = r \circ g \circ r'$ for some isomorphisms r and r' (see Remark 1.2.3).

Lemma 2.1.3. *Let* $\mathbf{C}$ *be a category with a binary operation* $*$. *There is a natural choice of binary operation* $*$ *on* $\mathbf{C}^{\to}$ *such that the functor* $F\colon \mathbf{C} \to \mathbf{C}^{\to}$ *given by* $a \mapsto \mathrm{id}_a$ *for all* $a \in \mathbf{C}$ *and* $f \mapsto (f, f)$ *for all* $f \in \operatorname{Mor}\mathbf{C}$ *preserves the binary operation. This functor is fully faithful and isomorphism reflecting.*

Proof. Let the action of $*$ on the objects of $\mathbf{C}^{\to}$ be given by the action of $*$ on the morphisms of $\mathbf{C}$ and let the action of $*$ on the morphisms of $\mathbf{C}^{\to}$ be given by $(g, g') * (f, f') = (g * f, g' * f')$.

We see that $F(a * b) = \mathrm{id}_{a*b} = \mathrm{id}_a * \mathrm{id}_b = \mathrm{id}_a * \mathrm{id}_b = F(a) * F(b)$ and $F(g * f) = (g * f, g * f) = (g, g) * (f, f) = F(g) * F(f)$. Hence, F preserves the binary operation.

It is easy to see that F is faithful. Suppose now that $(g, f) \in \operatorname{Hom}(F(a), F(b)) = \operatorname{Hom}(\mathrm{id}_a, \mathrm{id}_b)$. Then, $g = f$ and thus $(g, f) = F(f)$, so F is full.

Now, suppose $a \cong b$. Then, there exists an isomorphism $h\colon a \to b$. Since $\mathrm{id}_a = h^{-1} \circ \mathrm{id}_b \circ h$, we have $F(a) \cong F(b)$ and hence, F is isomorphism reflecting. $\square$

Remark 2.1.4. Note that the functor in the previous lemma is not necessarily an equivalence of categories. This is because F is not necessarily essentially surjective. To see this, let F be essentially surjective and suppose that f is a morphism that is not an isomorphism. Then, there exists an object a such that $F(a) \cong f$. Thus, we have $\mathrm{id}_a \cong f$ and so $f = g \circ \mathrm{id}_a \circ h$, where g and h are isomorphisms. Since composition of isomorphisms are again isomorphisms, f is an isomorphism, which is a contradiction.

Lemma 2.1.5. *If* $*$ *is a binary operation on a category* $\mathbf{C}$, *then the following hold:*

(a) For any objects $a, a', b, b' \in \operatorname{Ob}\mathbf{C}$, *we have*

$$a \cong a' \text{ and } b \cong b' \implies a * b \cong a' * b'. \tag{2.1.2}$$

(b) For any morphisms $f, f', g, g' \in \operatorname{Mor}\mathbf{C}$, *we have*

$$f \cong f' \text{ and } g \cong g' \implies f * g \cong f' * g'. \tag{2.1.3}$$

Proof. (a) If $f\colon a \to a'$ and $g\colon b \to b'$ are isomorphisms, then $f * g\colon a * b \to a' * b'$ is an isomorphism with $(f * g)^{-1} = f^{-1} * g^{-1}$.

(b) Suppose we have $f = a' \xrightarrow{k} a \xrightarrow{f'} b \xrightarrow{k'} b'$ and $g = c' \xrightarrow{r} c \xrightarrow{g'} d \xrightarrow{r'} d'$ where k, k', r, and r' are isomorphisms. Then, we have

$$f * g = (k' \circ f' \circ k) * (r' \circ g' \circ r) = (k' * r') \circ (f' * g') \circ (k * r).$$

Since $k' * r'$ and $k * r$ are both isomorphisms, the statement holds. $\square$

Example 2.1.6. Recall that a set X can be viewed as a discrete category $\mathbf{X}$ with $\operatorname{Ob}\mathbf{X} = X$. For every binary operation $*$ on a nonempty set X, there is a unique binary operation $*$ on $\mathbf{X}$ that has its action on objects defined by the action of $*$ on X. Since the only morphisms are the identity morphisms, the action of $*$ on morphisms is completely determined, namely by $(\mathrm{id}_a, \mathrm{id}_b) \mapsto \mathrm{id}_{a*b}$ for all pairs (a, b) of objects in $\mathbf{X}$.

By Example 2.1.6, a binary operation on a category can be thought of as a generalization of a binary operation on a set. The following notions extend to a binary operation on a category:

(a) We say $*$ is *associative* on $\mathbf{C}$ if $(a * b) * c \cong a * (b * c)$ for all objects $a, b, c \in \mathbf{C}$ and $f * (g * h) \cong (f * g) * h$ for all morphisms $f, g, h \in \operatorname{Mor}\mathbf{C}$.

(b) We say $*$ *has identity* if for some $e \in \mathbf{C}$, we have $e * a \cong a \cong a * e$ for all $a \in \mathbf{C}$ and $f * \mathrm{id}_e \cong f \cong \mathrm{id}_e * f$ for all $f \in \operatorname{Mor}\mathbf{C}$.

(c) We say $*$ is *commutative* on $\mathbf{C}$ if $a * b \cong b * a$ for all $a, b \in \mathbf{C}$ and $f * g \cong g * f$ for all $f, g \in \operatorname{Mor}\mathbf{C}$.

Example 2.1.7. Every (nonempty) category has a binary operation. Let $\mathbf{C}$ be some category and $e \in \mathbf{C}$ be an arbitrary object. Then, the map

$$\begin{aligned} *\colon \mathbf{C} \times \mathbf{C} &\to \mathbf{C}, \\ (a, b) &\mapsto e && \text{for all } a, b \in \mathbf{C}, \\ (f, g) &\mapsto \mathrm{id}_e && \text{for all } f, g \in \operatorname{Mor}\mathbf{C}, \end{aligned}$$

is a binary operation on $\mathbf{C}$. It is easy to see that $*$ is associative and commutative on $\mathbf{C}$. We call $*$ a *trivial binary operation* on $\mathbf{C}$.

Definition 2.1.8 ($S^{\mathrm{iso}}(\mathbf{C})$)**.** Let $\mathbf{C}$ be a category. We define

$$S^{\mathrm{iso}}(\mathbf{C}) := \operatorname{Ob}\mathbf{C}/\cong,$$

to be the class of all isomorphism classes of objects in $\mathbf{C}$. If $*$ is a binary operation on $\mathbf{C}$, we define a binary operation on $S^{\mathrm{iso}}(\mathbf{C})$ by $[a] * [b] = [a * b]$ and this is well defined by (2.1.2). The class $S^{\mathrm{iso}}(\mathbf{C})$ equipped with this operation is denoted by $(S^{\mathrm{iso}}(\mathbf{C}), *)$. Although we use the same symbol here, the intended operation will always be clear from the context.

Lemma 2.1.9. *Let $*$ be a binary operation on a category $\mathbf{C}$ and let $*$ be the corresponding binary operation on $\mathbf{C}^{\to}$ (see Lemma 2.1.3). Then, the map $f\colon (S^{\mathrm{iso}}(\mathbf{C}), *) \to (S^{\mathrm{iso}}(\mathbf{C}^{\to}), *)$ given by $f([a]) = [\mathrm{id}_a]$ preserves the binary operation.*

Proof. Note that $f([a]) = [F(a)]$ where F is the functor given in Lemma 2.1.3. The map is well defined since F is isomorphism reflecting. Since F preserves the binary operation, we have $f([a] * [b]) = f([a * b]) = [F(a * b)] = [F(a) * F(b)] = [F(a)] * [F(b)] = f([a]) * f([b])$. $\square$

Lemma 2.1.10. *Let $*$ be an binary operation on an essentially small category $\mathbf{C}$. If $*$ on $\mathbf{C}$ is associative, then $(S^{\text{iso}}(\mathbf{C}), *)$ and $(S^{\text{iso}}(\mathbf{C}^{\to}), *)$ are semigroups. If $*$ on $\mathbf{C}$ is associative and has identity, then $(S^{\text{iso}}(\mathbf{C}), *)$ and $(S^{\text{iso}}(\mathbf{C}^{\to}), *)$ are monoids. If $*$ is commutative on $\mathbf{C}$, then $(S^{\text{iso}}(\mathbf{C}), *)$ and $(S^{\text{iso}}(\mathbf{C}^{\to}), *)$ are commutative.*

Proof. $(S^{\text{iso}}(\mathbf{C}))$ is a semigroup because $([a]*[b])*[c] = [(a*b)*c] = [a*(b*c)] = [a]*([b]*[c])$. The proofs of the remaining statements are similar and are left to the reader. $\square$

In order to discuss the Grothendieck group with respect to a binary operation on a category, we must first restrict the categories we are working with to essentially small categories. This is because we need $S^{\text{iso}}(\mathbf{C})$ to be a set.

Definition 2.1.11 ($G^{\text{iso}}(\mathbf{C})$)**.** Let $\mathbf{C}$ be an essentially small category. The free abelian group generated by $S^{\text{iso}}(\mathbf{C})$ is denoted by $G^{\text{iso}}(\mathbf{C})$.

Example 2.1.12. Recall that two finite sets are isomorphic if and only if they have the same cardinality. Thus, if for each $n \in \mathbb{N}$ we fix a set X_n of cardinality n, then $G^{\text{iso}}(\mathbf{FinSet})$ is the free abelian group generated by $\{[X_0], [X_1], [X_2], \ldots\}$.

Definition 2.1.13 (Grothendieck group with respect to $*$)**.** Let $\mathbf{C}$ be an essentially small category and let $*$ be a binary operation on $\mathbf{C}$. Define $N(\mathbf{C}) \trianglelefteq G^{\text{iso}}(\mathbf{C})$ to be the (normal) subgroup generated by $\{[a * b] - [a] - [b] \mid a, b \in \operatorname{Ob}\mathbf{C}\}$. The *Grothendieck group with respect to* $*$, denoted $G_0(\mathbf{C}, *)$, is the quotient group $G^{\text{iso}}(\mathbf{C})/N(\mathbf{C})$. By a common abuse of terminology, we will write $[a]$ to mean the image of the isomorphism class $[a]$ in $G_0(\mathbf{C}, *)$ when this will not cause confusion. It follows that $[a*b] = [a]+[b]$ in $G_0(\mathbf{C}, *)$ for all $a, b \in \mathbf{C}$.

Example 2.1.14. A classic example is $G_0(\mathbf{Set}, \amalg)$. Recall that the binary coproduct in $\mathbf{Set}$ is given by the disjoint union:

$$X_1 \amalg X_2 = \{(x, i) \mid i \in \{1, 2\} \text{ and } x \in X_i\}$$

Since $|X \amalg Y| = |X| + |Y|$, the binary operation $(X, Y) \mapsto X \amalg Y$ satisfies (2.1.2). There is a natural choice of binary operation on $\operatorname{Mor}\mathbf{Set}$, namely for arbitrary morphisms $f \colon X \to Y$ and $g \colon X' \to Y'$ we map $(f, g) \mapsto f \amalg g$ where $f \amalg g \colon X \amalg X' \to Y \amalg Y'$ is given by

$$(f \amalg g)(x, i) = \begin{cases} f(x) \text{ if } i = 1 \\ g(x) \text{ if } i = 2 \end{cases} .$$

We leave the verification that this binary operation satisfies (2.1.3) to the reader. The group $G_0(\mathbf{Set}, \amalg)$ is isomorphic to $\mathbb{Z}$ by the First Isomorphism Theorem—cardinality induces a surjective group homomorphism from $G^{\text{iso}}(\mathbf{Set})$ to $\mathbb{Z}$ whose kernel coincides with $N(\mathbf{Set})$. As first discussed in the introduction, this is the process of *decategorification* – that of reducing structures on categories to set-theoretic structures. This example is considered classic because, according to some mathematicians, it recovers the integers in a way that parallels how the human brain initially understands the integers.

Lemma 2.1.15. *If $*$ has identity e on $\mathbf{C}$, then $[e] = 0$ in $G_0(\mathbf{C}, *)$.*

Proof. Since $[e] = -[e] + [e] + [e] = -([e * e] - [e] - [e]) \in N(\mathbf{C})$, we have $[e] = 0$ in $G_0(\mathbf{C}, *)$. $\square$

Lemma 2.1.16. *Let $*$ be an associative binary operation on an essentially small category $\mathbf{C}$. Then, in $G_0(\mathbf{C}, *)$ we have the following:*

(a) Every element can be written in the form $[a] - [b]$ for some $a, b \in \mathbf{C}$,

(b) We have $[a] = [b]$ if and only if $a * c \cong b * c$ for some $c \in \mathbf{C}$.

Proof. (a) Suppose x is an element in $G_0(\mathbf{C}, *)$. Since the addition in $G_0(\mathbf{C}, *)$ is commutative, x can be written in the form $\alpha_1[a_1] + \cdots + \alpha_n[a_n] - \beta_1[b_1] - \cdots - \beta_m[b_m]$ where $n, m \in \mathbb{N}_+$ and $\alpha_i, \beta_i > 0$ for all i. We thus have

$$x = \left[\mathop{*}_{i=1}^{n} \mathop{*}_{j=1}^{\alpha_i} a_i\right] - \left[\mathop{*}_{i=1}^{m} \mathop{*}_{j=1}^{\beta_i} b_i\right].$$

(b) The reverse implication holds since $[a] + [c] = [a * c] = [b * c] = [b] + [c]$. For the forward implication, if $[a] = [b]$ in $G_0(\mathbf{C}, *)$, then $[a] - [b] \in N(\mathbf{C})$. So, in $G^{\text{iso}}(\mathbf{C})$, $[a] - [b]$ is of the form $\sum_{i=1}^{n}([x_i * y_i] - [x_i] - [y_i]) - \sum_{i=1}^{m}([w_i * z_i] - [w_i] - [z_i])$ for some $n, m \in \mathbb{N}_+$. Rearranging the terms, we obtain

$$[a] + \sum_{i=1}^{m}[w_i * z_i] + \sum_{i=1}^{n}([x_i] + [y_i]) = [b] + \sum_{i=1}^{n}[x_i * y_i] + \sum_{i=1}^{m}([w_i] + [z_i]).$$

Since the isomorphism classes are a basis of $G^{\text{iso}}(\mathbf{C})$, the terms on one side of the equation are a permutation of the terms on the other. Since (2.1.2) holds, we have $a * c \cong b * c$, where

$$c = \left(\mathop{*}_{i=1}^{n}(x_i * y_i)\right) * \left(\mathop{*}_{i=1}^{m}(w_i * z_i)\right). \qquad \square$$

Proposition 2.1.17 (Group completion of a commutative semigroup)**.** *Let $(M, *)$ be a commutative semigroup. There exists a unique (up to isomorphism)* group completion *which is a pair (G, φ) where G is an abelian group and $\varphi \colon M \to G$ is a semigroup homomorphism satisfying the universal property that for all semigroup homomorphisms $\psi \colon M \to H$ where $(H, +)$ is some (not necessarily abelian) group, there exists a unique group homomorphism $\theta \colon G \to H$ such that $\psi = \theta \circ \varphi$. This statement can be summarized with the following commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\varphi"'] & H \\ G \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

Proof. See [21, p. 4] for a proof of this statement. $\square$

Theorem 2.1.18. *Let* $\mathbf{C}$ *be a category and let* $*$ *be an associative and commutative binary operation on* $\mathbf{C}$*. Then, the Grothendieck group with respect to* $*$ *is a group completion of the semigroup* $(S^{\text{iso}}(\mathbf{C}), *)$*.*

Proof. Let $\varphi\colon S^{\text{iso}}(\mathbf{C}) \to G_0(\mathbf{C}, *)$ be the natural map $[a] \mapsto [a]$. Let $\psi\colon S^{\text{iso}}(\mathbf{C}) \to H$ be some semigroup homomorphism where H is a (not necessarily abelian) group. We claim that the group homomorphism

$$\begin{aligned} \theta\colon G_0(\mathbf{C}, *) &\to H, \\ [c] &\mapsto \psi([c]), \end{aligned}$$

extended by linearity is the unique morphism such that $\psi = \theta \circ \varphi$. Since ψ is defined on $S^{\text{iso}}(\mathbf{C})$, it is clear that θ is well defined on $G^{\text{iso}}(\mathbf{C})$. For any objects a and b, we have

$$\begin{aligned} \theta([a * b] - [a] - [b]) &= \psi([a * b]) \cdot \psi([a])^{-1} \cdot \psi([b])^{-1} \\ &= \psi([b * a]) \cdot \psi([a])^{-1} \cdot \psi([b])^{-1} \\ &= \psi([b]) \cdot \psi([a]) \cdot \psi([a])^{-1} \cdot \psi([b])^{-1} \\ &= 0 \end{aligned}$$

and, thus, θ is well defined on $G_0(\mathbf{C}, *)$. Note that the commutativity of $*$ is necessary since we did not assume that H is abelian. Suppose $\sigma\colon G_0(\mathbf{C}, *) \to H$ is a group homomorphism that also satisfies $\psi = \sigma \circ \varphi$. Then we have

$$\sigma([c]) = \sigma(\varphi([c])) = \psi([c]) = \theta([c])$$

and, thus $\sigma = \theta$. □

Corollary 2.1.19. *Let* $\mathbf{Cat}_*$ *be the category of all pairs consisting of a category and a binary operation (on that category). Morphisms in this category are functors between the categories that preserve the binary operation. The Grothendieck group with respect to a binary operation induces the following functor:*

$$\begin{aligned} G_0\colon \mathbf{Cat}_* &\to \mathbf{Ab}, \\ (\mathbf{C}, *) &\mapsto G_0(\mathbf{C}, *) && \textit{for all } (\mathbf{C}, *) \in \mathbf{Cat}_*, \\ F &\mapsto (a \mapsto [F(a)]) && \textit{for all } F \in \operatorname{Mor} \mathbf{Cat}_*. \end{aligned}$$

We now have two good ways of verifying the structure of $G_0(\mathbf{C}, *)$ where $*$ is an associative and commutative binary operation on a category $\mathbf{C}$. Suppose we suspect that $G_0(\mathbf{C}, *) \cong G$. One way to show it is to look for a surjective group homomorphism $G^{\text{iso}}(\mathbf{C}) \to G$ whose kernel coincides with $N(\mathbf{C})$ and, thus, conclude that $G_0(\mathbf{C}, *) \cong G$ by the First Isomorphism Theorem. Alternatively, we can show that G is a group completion of $(S^{\text{iso}}(\mathbf{C}), *)$ and then conclude $G_0(\mathbf{C}, *) \cong G$ by Theorem 2.1.18.

Lemma 2.1.20. *Let* $\mathbf{C}$ *be a category and let* $*$ *be a binary operation on* $\mathbf{C}$*. Suppose there is an object* c *such that* $c * a \cong c$ *for all objects* $a \in \mathbf{C}$*. Then* $G_0(\mathbf{C}, *)$ *is the trivial group.*

Proof. Since $[a] = [c] - [c] + [a] = [c * a] - [c] - [a] \in N(\mathbf{C})$, the Grothendieck group must be trivial. $\square$

Example 2.1.21. Suppose we have two binary operations $+$ and $\cdot$ on a category $\mathbf{C}$ such that $(S^{\text{iso}}(\mathbf{C}), +, \cdot)$ is a ring. Let 0 be a representative object in $\mathbf{C}$ for the additive identity of $S^{\text{iso}}(\mathbf{C})$. By Lemma 2.1.20, we have that $G_0(\mathbf{C}, \cdot)$ is the trivial group since $0 \cdot a \cong 0$ for all objects a.

2.2 Additive Categories

Additive categories form the basic building block for the remainder of the categories discussed in this chapter. We will see that, in additive categories, binary products and coproducts coincide, and we will use this operation to define the *split Grothendieck group* of an additive category.

Definition 2.2.1 (Preadditive/additive category)**.** A category $\mathbf{C}$ is *additive* if the following conditions hold:

(AD1) Every hom-class has the structure of an abelian group (an addition), denoted by $+$ unless stated otherwise, and composition is distributive over this addition. By this, we mean for any morphisms $f\colon a \to b$, $f'\colon a \to b$, $g\colon b \to c$, and $g'\colon b \to c$ we have:

- (a) $(g + g') \circ f = g \circ f + g' \circ f$,
- (b) $g \circ (f + f') = g \circ f + g \circ f'$,

(AD2) The category $\mathbf{C}$ has a zero object.

(AD3) The binary product and the binary coproduct of any two objects exists.

We will typically emphasize that a category is additive by opting to denote the category by $\mathbf{A}$. Categories where (AD1) holds are called *preadditive*.

Example 2.2.2. $\mathbf{Ab}$, $\mathbf{Vect}_{\mathbb{K}}$, and R-$\mathbf{Mod}$ are additive categories for any field $\mathbb{K}$ and any ring R.

Remark 2.2.3. It follows from (AD1) that any preadditive category has zero morphisms, and the zero morphisms are the identities of the hom-classes. Furthermore, for any $f\colon a \to b$, $g\colon b \to c$, and $z \in \mathbb{Z}$ we have:

$$(zg) \circ f = g \circ (zf) = z(g \circ f).$$

Therefore, the addition of the hom-classes is bilinear over $\mathbb{Z}$ with respect to composition.

Example 2.2.4. A simple example of a category that is not preadditive is $\mathbf{Set}$. For instance, $\mathrm{Hom}(\{1,2\},\{1,2\}) = \{f_1, f_2, \mathrm{id}, r\}$, where f_1 and f_2 are the respective constant functions and r is the function that maps each number to the other. Hence, the hom-class has the following composition:

$\circ$	f_1	f_2	id	r
f_1	f_1	f_1	f_1	f_1
f_2	f_2	f_2	f_2	f_2
id	f_1	f_2	id	r
r	f_2	f_1	r	id

Furthermore, if this hom-class had an addition bilinear with respect to composition, then $r \circ (f_1 + f_2) = r \circ f_1 + r \circ f_2 = f_2 + f_1 = f_1 + f_2$ and, by the above composition structure, no choice of morphism for $f_1 + f_2$ satisfies this equation.

Lemma 2.2.5. *Let* $\mathbf{A}$ *be an additive category and* $f\colon b \to c \in \operatorname{Mor} \mathbf{A}$*. The following are equivalent:*

(a) The morphism f *is monic.*

(b) For any object a *and morphism* $g\colon a \to b$ *the following holds:*

$$f \circ g = 0_{ac} \implies g = 0_{ab}.$$

Proof. Suppose f satisfies (b), and for some morphisms $h\colon d \to b$ and $h'\colon d \to b$, we have $f \circ h = f \circ h'$. Then, $f \circ (h - h') = 0_{dc}$ so $h - h' = 0_{db}$ and hence, $h = h'$. Thus, f is a monomorphism. The other implication was proved in Lemma 1.2.19. $\square$

Definition 2.2.6 (Binary biproduct)**.** Let $\mathbf{C}$ be a category that has a zero object 0. An object is a *binary biproduct of* a *and* b, denoted $a \oplus b$, if there are morphisms $p_a\colon a \oplus b \to a$ and $p_b\colon a \oplus b \to b$ that make $a \oplus b$ a product of a and b and morphisms $i_a\colon a \to a \oplus b$ and $i_b\colon b \to a \oplus b$ that make $a \oplus b$ a coproduct of a and b that, in addition, satisfy the following equations:

$$p_a \circ i_a = \mathrm{id}_a, \quad p_b \circ i_b = \mathrm{id}_b, \quad p_b \circ i_a = 0_{a,b}, \quad \text{and} \quad p_a \circ i_b = 0_{b,a}. \tag{2.2.1}$$

To emphasize the morphisms, we will sometimes denote the biproduct of a and b as a quintuple $(a \oplus b, p_a, p_b, i_a, i_b)$.

Remark 2.2.7. Biproducts are unique up to isomorphism because they are a product (and products are unique up to isomorphism).

Proposition 2.2.8. *Let* $\mathbf{A}$ *be an additive category. Then, the following hold:*

(a) Binary products and coproducts are equivalent and are biproducts.

(b) A binary biproduct of two objects a *and* b *satisfies an additional equation:*

$$i_a \circ p_a + i_b \circ p_b = \mathrm{id}_{a \oplus b}\,.$$

Proof. (a) Let $a \coprod b$ be a binary coproduct of objects a and b, and fix morphisms $i_a \colon a \to a \coprod b$ and $i_b \colon b \to a \coprod b$ as in the definition of a coproduct. Let $p_a \colon a \coprod b \to a$ and $p_b \colon a \coprod b \to b$ be the unique morphisms that make the following diagrams commute:

$$
\begin{tikzcd}
 & a & \\
a \arrow[ur, "\mathrm{id}_a"] \arrow[r, "i_a"] & a \coprod b \arrow[u, dotted, "p_a"'] & b \arrow[l, "i_b"'] \arrow[ul, "0_{b,a}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & b & \\
a \arrow[ur, "0_{a,b}"] \arrow[r, "i_a"] & a \coprod b \arrow[u, dotted, "p_b"'] & b \arrow[l, "i_b"'] \arrow[ul, "\mathrm{id}_b"']
\end{tikzcd}
\tag{2.2.2}
$$

Let $f_a \colon c \to a$ and $f_b \colon c \to b$ be arbitrary morphisms. Then, the morphism $f = i_a \circ f_a + i_b \circ f_b$ makes the following diagram commute:

$$
\begin{tikzcd}
 & c \arrow[dl, "f_a"'] \arrow[d, dotted, "f"] \arrow[dr, "f_b"] & \\
a & a \coprod b \arrow[l, "p_a"'] \arrow[r, "p_b"] & b
\end{tikzcd}
$$

Suppose $f' \colon c \to a \coprod b$ also satisfies $f_a = p_a \circ f'$ and $f_b = p_b \circ f'$. Then,

$$f = i_a \circ (p_a \circ f') + i_b \circ (p_b \circ f') = (i_a \circ p_a + i_b \circ p_b) \circ f'.$$

Let $u = i_a \circ p_a + i_b \circ p_b$. Note that $\mathrm{id}_{a \coprod b}$ also makes the following diagram commute in place of u:

$$
\begin{tikzcd}
 & a \coprod b & \\
a \arrow[ur, "i_a"] \arrow[r, "i_a"] & a \coprod b \arrow[u, dotted, "u"'] & b \arrow[l, "i_b"'] \arrow[ul, "i_b"']
\end{tikzcd}
$$

By the uniqueness of such a morphism in the definition of coproduct, we have $u = \mathrm{id}_{a \coprod b}$ and hence $f = f'$. Thus, we have shown that $a \coprod b$ is a product of a and b. Since there is a coproduct for every pair of objects in an additive category, every coproduct is a product, and both coproducts and products are unique up to isomorphism, we conclude that coproducts and products are equivalent. Additionally, the equations that make $a \coprod b$ a biproduct are given by the commutativity of (2.2.2).

(b) Since $a \oplus b$ is both a coproduct and a product of a and b, the exact same argument as in part (a) shows that $u = \mathrm{id}_{a \coprod b}$. $\square$

Proposition 2.2.9. *Let $\mathbf{A}$ be an additive category. For any objects $a, b, c \in \mathbf{A}$, the following are equivalent:*

(a) $c \cong a \oplus b$.

(b) There are morphisms $i_a \colon a \to c$, $i_b \colon b \to c$, $p_a \colon c \to a$, and $p_b \colon c \to b$ that satisfy the following equations:

$$
\begin{gathered}
p_a \circ i_a = \mathrm{id}_a, \quad p_b \circ i_b = \mathrm{id}_b, \quad p_a \circ i_b = 0_{ba}, \\
p_b \circ i_a = 0_{ab}, \quad \text{and} \quad i_a \circ p_a + i_b \circ p_b = \mathrm{id}_{a \oplus b}\,.
\end{gathered}
\tag{2.2.3}
$$

Proof. The forward implication holds since biproducts are unique up to isomorphism and, thus, the equations in (2.2.3) are satisfied by definition and by Proposition 2.2.8(b). For the other direction, by Proposition 2.2.8(a), it suffices to show that that c is a coproduct of a and b. Suppose that (2.2.3) holds. For any morphisms $f_a \colon a \to d$ and $f_b \colon b \to d$, the morphism $f = f_a \circ p_a + f_b \circ p_b$ clearly satisfies both $f \circ i_a = f_a$ and $f \circ i_b = f_b$. It remains to show that f is unique. Suppose there is an $f' \colon a \oplus b \to d$ such that $f' \circ i_a = f_a$ and $f' \circ i_b = f_b$. Then,

$$f = (f' \circ i_a) \circ p_a + (f' \circ i_b) \circ p_b = f' \circ (i_a \circ p_a + i_b \circ p_b) = f' \circ \mathrm{id}_{a \oplus b} = f'.$$

Hence, c is a coproduct of a and b and, thus, is a biproduct of a and b. $\square$

Remark 2.2.10. The biproduct is not always cancellative. That is, if $a \oplus b \cong a \oplus c$, then we do not necessarily have $b \cong c$. For example, consider the additive category $\mathbb{Z}$-**Mod**. Let $M = \bigoplus_{i=1}^{\infty} \mathbb{Z}$. We have $M \oplus \mathbb{Z} \cong M \cong M \oplus \{0\}$, but $\mathbb{Z} \not\cong \{0\}$.

Definition 2.2.11 (Diagonal morphism, codiagonal morphism, biproduct morphism)**.** Let $\mathbf{C}$ be an additive category and a be any object. Fix morphisms for i_a, i'_a, p_a, and p'_a as in the definition of the biproduct $a \oplus a$. The morphisms $\triangle_a \colon a \to a \oplus a$, the *diagonal morphism*, and $\nabla_a \colon a \oplus a \to a$, the *codiagonal morphism*, are defined to be the unique morphisms that make the following diagrams, respectively, commute:

$$\begin{tikzcd} & a \arrow[dl, "\mathrm{id}_a"'] \arrow[d, dotted, "\triangle_a"] \arrow[dr, "\mathrm{id}_a"] & \\ a & a \oplus a \arrow[l, "p_a"'] \arrow[r, "p'_a"] & a \end{tikzcd} \qquad \begin{tikzcd} & a & \\ a \arrow[ur, "\mathrm{id}_a"] \arrow[r, "i_a"'] & a \oplus a \arrow[u, dotted, "\nabla_a"'] & a \arrow[ul, "\mathrm{id}_a"'] \arrow[l, "i'_a"] \end{tikzcd}$$

Let $a \oplus c$ and $b \oplus d$ be biproducts, with morphisms denoted in the usual way. The *biproduct morphism* of $f \colon a \to b$ and $g \colon c \to d$ is the unique morphism $f \oplus g \colon a \oplus c \to b \oplus d$ satisfying

$$p_b \circ (f \oplus g) \circ i_a = f, \quad p_d \circ (f \oplus g) \circ i_c = g, \quad p_b \circ (f \oplus g) \circ i_c = 0_{c,b}, \quad \text{and} \quad p_d \circ (f \oplus g) \circ i_a = 0_{a,d}.$$

Such a morphism exists as one can take

$$f \oplus g = i_b \circ f \circ p_a + i_d \circ g \circ p_c. \tag{2.2.4}$$

To show uniqueness, first consider the following two commutative diagrams:

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[d, dotted, "(f\oplus g)\circ i_a"] \arrow[dr, "0_{a,d}"] & \\ b & b \oplus d \arrow[l, "p_b"'] \arrow[r, "p_d"] & d \end{tikzcd} \qquad \begin{tikzcd} & c \arrow[dl, "0_{c,b}"'] \arrow[d, dotted, "(f\oplus g)\circ i_c"] \arrow[dr, "g"] & \\ b & b \oplus d \arrow[l, "p_b"'] \arrow[r, "p_d"] & d \end{tikzcd}$$

Hence, the morphisms $(f \oplus g) \circ i_a$ and $(f \oplus g) \circ i_c$ are unique by the definition of product. By the definition of coproduct, $(f \oplus g)$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd} & b \oplus d & \\ a \arrow[ur, "(f\oplus g)\circ i_a"] \arrow[r, "i_a"'] & a \oplus c \arrow[u, dotted, "f\oplus g"'] & c \arrow[ul, "(f\oplus g)\circ i_c"'] \arrow[l, "i_c"] \end{tikzcd}$$

Proposition 2.2.12. *In an additive category, the addition (that is bilinear with respect to composition) of a hom-class* $\operatorname{Hom}(a,b)$ *is unique and given by*

$$f+g=\nabla_b\circ(f\oplus g)\circ\triangle_a.$$

Proof. Fix an addition $+$ on the hom-sets and let i_a, i_a', p_a, and p_a' be as in Definition 2.2.11. By uniqueness, we have $\triangle_a=i_a+i_a'$. Similarly, we have $\nabla_b=p_b+p_b'$. Hence,

$$\nabla_b\circ(f\oplus g)\circ\triangle_a=(p_b+p_b')\circ(f\oplus g)\circ(i_a+i_a')=f+0_{a,b}+0_{a,b}+g=f+g$$

Fix another addition $+'$ on the hom-sets, but keep i_a, i_a', p_a, and p_a' as before. By uniqueness, we have $\triangle_a=i_a+'i_a'$ and $\nabla_b=p_b+'p_b'$. Hence,

$$f+g=\nabla_b\circ(f\oplus g)\circ\triangle_a=f+'g$$

We conclude the addition on every hom-class $\operatorname{Hom}(a,b)$ is unique. □

Lemma 2.2.13. *Let* $\mathbf{A}$ *be an additive category. Fix a biproduct for every two objects. The functor*

$$\begin{aligned}\oplus\colon \mathbf{A}\times\mathbf{A}&\to\mathbf{A}\\ (a,b)&\mapsto a\oplus b &&\text{for all } a,b\in\mathbf{A}\\ (f,g)&\mapsto f\oplus g &&\text{for all } f,g\in\operatorname{Mor}\mathbf{A}\end{aligned}$$

is well defined, and thus $\oplus$ *is a binary operation on* $\mathbf{A}$*.*

Proof. The biproduct of any two objects is guaranteed by (AD3). We noted in the definition of biproduct morphism that the biproduct of any two morphisms exists. Hence, it remains to check that the functor preserves identity morphisms and composition. Let $a,b\in\mathbf{A}$ and fix a biproduct $(a\oplus b,i_a,i_b,p_a,p_b)$ of a and b. By writing biproducts in the form (2.2.4) and then using Proposition 2.2.8(b), we have

$$\mathrm{id}_a\oplus\mathrm{id}_b=i_a\circ\mathrm{id}_a\circ p_a+i_b\circ\mathrm{id}_b\circ p_b=\mathrm{id}_{a\oplus b}.$$

Thus, the map $\oplus$ preserves identity morphisms. Now, let there be morphisms $a\xrightarrow{f}b\xrightarrow{g}c$ and $a'\xrightarrow{f'}b'\xrightarrow{g'}c'$, and fix biproducts $(a\oplus a',i_a,i_{a'},p_a,p_{a'})$, $(b\oplus b',i_b,i_{b'},p_b,p_{b'})$, and $(c\oplus c',i_c,i_{c'},p_c,p_{c'})$ of the obvious objects. By again writing biproducts in the form (2.2.4), we see that

$$\begin{aligned}(g\circ f)\oplus(g'\circ f')&=i_c\circ g\circ f\circ p_a+i_{c'}\circ g'\circ f'\circ p_{a'}\\&=(i_c\circ g\circ p_b+i_{c'}\circ g'\circ p_{b'})\circ(i_b\circ f\circ p_a+i_{b'}\circ f'\circ p_{a'})\\&=(g\oplus g')\circ(f\oplus f').\end{aligned}$$

Hence, the map $\oplus$ preserves composition and is, therefore, a binary operation on $\mathbf{A}$. □

Proposition 2.2.14. *Let* $\mathbf{A}$ *be an additive category and let* $\oplus$ *be the binary operation on* $\mathbf{A}$ *given in Lemma 2.2.13. Both* $(S^{\mathrm{iso}}(\mathbf{A}),\oplus)$ *and* $(S^{\mathrm{iso}}(\mathbf{A}^{\to}),\oplus)$ *are commutative monoids.*

Proof. By Lemma 1.3.6, we have that $\oplus$ is associative and commutative up to isomorphism on $\operatorname{Ob}\mathbf{A}$ (since the biproduct is a product). Let $(a \oplus b, p_a, p_b, i_a, i_b)$, $(b \oplus a, p'_b, p'_a, i'_b, i'_a)$, $(c \oplus d, p_c, p_d, i_c, i_d)$, and $(d \oplus c, p'_d, p'_c, i'_d, i'_c)$ be the fixed biproducts of the binary operation. By writing $f \oplus g\colon a \oplus c \to b \oplus d$ in the form (2.2.4), we note that

$$i_c \circ f \circ p_a + i_d \circ g \circ p_b = (i_d \circ p'_d + i_c \circ p'_c) \circ (i'_d \circ g \circ p'_b + i'_c \circ f \circ p'_a) \circ (i'_b \circ p_b + i'_a \circ p_a)$$

and, hence, $f \oplus g \cong g \oplus f$ since $i_d \circ p'_d + i_c \circ p'_c$ and $i'_b \circ p_b + i'_a \circ p_a$ are isomorphisms. The associativity of the morphisms up to isomorphism is proved in a similar manner and is left to the reader. Hence, $\oplus$ is associative and commutative on $\mathbf{A}$. By Lemma 1.3.4, we have that $\oplus$ has identity 0 since $f \oplus \mathrm{id}_0 = i_a \circ f \circ p_a \cong f$. Hence, by Lemma 2.1.10, $(S^{\mathrm{iso}}(\mathbf{C}), \oplus)$ and $(S^{\mathrm{iso}}(\mathbf{A}^{\to}), \oplus)$ are commutative monoids. Note that this monoid is independent of our choice of biproducts since biproducts are unique up to isomorphism. □

Definition 2.2.15 (Split Grothendieck group)**.** The Grothendieck group with respect to $\oplus$ of an additive category $\mathbf{A}$, which we denote $G_0(\mathbf{A}, \oplus)$, is called the *split Grothendieck group* of $\mathbf{A}$. The subgroup generated by $\{[a \oplus b] - [a] - [b] \mid a, b \in \mathbf{A}\}$ is denoted by $N_{\oplus}(\mathbf{A})$.

Remark 2.2.16. By Lemma 2.1.15, $[0] = 0$ in $G_0(\mathbf{A}, \oplus)$.

Example 2.2.17. The split Grothendieck group of $\mathbf{FinVect}_{\mathbb{C}}$ is isomorphic as a group to $\mathbb{Z}$. Consider the homomorphism $f\colon G^{\mathrm{iso}}(\mathbf{FinVect}_{\mathbb{C}}) \to \mathbb{Z}$ given by $f([V]) = \dim(V)$ (and then extended by linearity). This is well defined as dimension is unique up to isomorphism. Since dimension is additive (i.e. $\dim(V \oplus W) = \dim(V) + \dim(W)$), we have $N_{\oplus}(\mathbf{FinVect}_{\mathbb{C}}) \subseteq \ker(f)$. Now, let $\sum_{i=1}^{n} \alpha_i [V_i]$ be arbitrary in $\ker(f)$. We have $\sum_{i=1}^{n} \alpha_i \dim(V_i) = f(\sum_{i=1}^{n} \alpha_i [V_i]) = 0$. In $G_0(\mathbf{FinVect}_{\mathbb{C}}, \oplus)$, since $[V_i] = \dim(V_i)[\mathbb{C}]$, we have $\sum_{i=1}^{n} \alpha_i [V_i] = (\sum_{i=1}^{n} \alpha_i \dim(V_i))[\mathbb{C}] = 0$, so $\ker(f) = N_{\oplus}(\mathbf{FinVect}_{\mathbb{C}})$. By the First Isomorphism Theorem,

$$G_0(\mathbf{FinVect}_{\mathbb{C}}, \oplus) \cong G^{\mathrm{iso}}(\mathbf{FinVect}_{\mathbb{C}}) / \ker(f) \cong \mathbb{Z}.$$

The following example illustrates why we usually only consider categories whose objects consist of the “finite” or “finitely generated” type.

Example 2.2.18. Let $\mathbb{K}$ be a field. Consider the category $\mathbf{Vect}_{\mathbb{K}}$. Let $V \in \mathbf{Vect}_{\mathbb{K}}$. Let $B = \bigoplus_{i=1}^{\infty} V$. In $G_0(\mathbf{Vect}_{\mathbb{K}}, \oplus)$, we have $[V] + [B] = [V \oplus B] = [B]$ and, hence, $[V] = 0$. Therefore, the split Grothendieck group of $\mathbf{Vect}_{\mathbb{K}}$ is trivial.

Definition 2.2.19 (Additive functor)**.** A functor $F\colon \mathbf{C} \to \mathbf{D}$ where $\mathbf{C}$ and $\mathbf{D}$ are preadditive categories is said to be a *additive functor* if it acts as a group homomorphism on the hom-classes with respect to the addition.

Lemma 2.2.20. *Let $\mathbf{C}$ and $\mathbf{D}$ be additive categories. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is additive if and only if it preserves biproducts. By preserving biproducts, we mean $F(a \oplus b) \cong F(a) \oplus F(b)$ for any $a, b \in \mathbf{C}$.*

Proof. The equations (2.2.3) are preserved by an additive functor (since an additive functor preserves composition and addition of morphisms as well as the identity and zero morphisms) and, thus, an additive functor preserves biproducts. For the reverse implication, we refer the reader to [5, Proposition 1.3.4]. $\square$

Proposition 2.2.21. *Let $\mathbf{A}$ and $\mathbf{A}'$ be additive categories. An additive functor $F\colon \mathbf{A} \to \mathbf{A}'$ induces a group homomorphism $[F]\colon G_0(\mathbf{A}, \oplus) \to G_0(\mathbf{A}', \oplus)$ such that $[F]([a]) = [F(a)]$ for any object $a \in \mathbf{A}$ and extended by linearity.*

Proof. Recall the generating set of $N_\oplus(\mathbf{A})$ defined by $\{[a \oplus b] - [a] - [b] \mid a, b \in \mathbf{A}\}$. Note that $[F]$ sends each element of this generating set to zero:

$$[F]([a \oplus b] - [a] - [b]) = [F(a \oplus b)] - [F(a)] - [F(b)] = [F(a) \oplus F(b)] - [F(a)] - [F(b)] = 0.$$

Hence, if $x = y$ in $G_0(\mathbf{A}, \oplus)$, then $x - y \in N_\oplus(\mathbf{A})$ so $[F](x - y) = 0$. Hence, $[F](x) = [F](y)$ and so $[F]$ is well defined. $\square$

Lemma 2.2.22. *If $\mathbf{A}$ is an additive category and $x \in \mathbf{A}$, then the functor $\mathrm{Hom}(x, -)$ given in Example 1.5.18 can be defined to have codomain $\mathbf{Ab}$. Furthermore, this functor is additive.*

Proof. For every $a \in \mathbf{A}$, the object $\mathrm{Hom}(x, a)$ is a group by (AD1). For every morphism $f\colon a \to b$, the morphism $\mathrm{Hom}(x, f)\colon \mathrm{Hom}(x, a) \to \mathrm{Hom}(x, b)$ is a group homomorphism, as we see that

$$\mathrm{Hom}(x, f)(k + k') = f \circ (k + k') = f \circ k + f \circ k' = \mathrm{Hom}(x, f)(k) + \mathrm{Hom}(x, f)(k').$$

The proof that the functor is additive is similar:

$$\mathrm{Hom}(x, f + f')(k) = (f + f') \circ k = f \circ k + f' \circ k = \mathrm{Hom}(x, f) + \mathrm{Hom}(x, f'). \qquad \square$$

From now on if $\mathbf{A}$ is an additive category, we will assume that for any $x \in \mathbf{A}$, the codomain of $\mathrm{Hom}(x, -)$ is $\mathbf{Ab}$ and not $\mathbf{Set}$.

Example 2.2.23. If an additive functor is an equivalence of categories, then it is both isomorphism reflecting and essentially surjective. However, if an additive functor is both isomorphism reflecting and essentially surjective, then it is not necessarily an equivalence of categories. Let R be a ring. The category $\mathbf{Mat}(R)$ is given by the following information:

$$\begin{aligned} \mathrm{Ob} &= \mathbb{N}, \\ \mathrm{Hom}(m, n) &= \{m \times n \text{ matrices with entries in } R, \text{ and } m, n \in \mathbb{N}\}, \\ X \circ Y &= YX. \end{aligned}$$

We use the convention that a matrix with 0 columns or 0 rows is empty and acts like a zero morphism. This category is additive with zero object 0. The addition on a hom-class is given by the usual addition of matrices. The biproduct of two objects is just the sum of the corresponding natural numbers. Consider the functor:

$$\begin{aligned} F\colon \mathbf{Mat}(\mathbb{Z}) \to& \mathbf{Mat}(\mathbb{Z}_2), \\ n \mapsto& n, \\ [a_{i,j}] \mapsto& [\overline{a_{i,j}}]. \end{aligned}$$

This additive functor is both isomorphism reflecting and essentially surjective, but not an equivalence of categories.

Definition 2.2.24 (Additive completion)**.** Let $\mathbf{C}$ be a preadditive category. The *additive completion* of $\mathbf{C}$, denoted $\mathbf{C}^+$, is a category given by the following:

(a) Objects in $\mathbf{C}^+$ are ordered tuples of objects in $\mathbf{C}$:

$$\operatorname{Ob}\mathbf{C}^+ = \{(a_1, \dots, a_n) \mid n \in \mathbb{N},\ a_i \in \operatorname{Ob}\mathbf{C} \text{ for all } i\}.$$

By a common convention, the case where $n = 0$ is called the *empty tuple* and is denoted $\varnothing$. In this setting, $\varnothing$ will serve as a zero object.

(b) Hom-classes are given by

$$\begin{aligned} \operatorname{Hom}_{\mathbf{C}^+}((a_1, \dots, a_n), (b_1, \dots, b_m)) &= \{[f_{i,j}] \mid f_{i,j} \in \operatorname{Hom}_{\mathbf{C}}(a_j, b_i), 1 \le i \le m, 1 \le j \le n\}, \\ \operatorname{Hom}_{\mathbf{C}^+}(\varnothing, x) &= \{0_{\varnothing,x}\} \text{ for all } x \in \operatorname{Ob}\mathbf{C}^+, \\ \operatorname{Hom}_{\mathbf{C}^+}(x, \varnothing) &= \{0_{x,\varnothing}\} \text{ for all } x \in \operatorname{Ob}\mathbf{C}^+. \end{aligned}$$

(c) Composition is given by $X \circ Y = YX$, where YX denotes the usual matrix product in the following sense: if $X = [f_{i,j}]$ ($1 \le i \le m$ and $1 \le j \le n$) and $Y = [g_{i,j}]$ ($1 \le i \le n$ and $1 \le j \le m'$) are composable, then

$$[(XY)_{i,j}] = \left[\sum_{k=1}^{n} (f_{i,k} \circ g_{k,j})\right].$$

Remark 2.2.25. If $\mathbf{C}$ is a preadditive category, then its additive completion is an additive category. The biproduct is given by $(a_1, \dots, a_n) \oplus (b_1, \dots, b_m) = (a_1, \dots, a_n, b_1, \dots, b_m)$. The zero object is $\varnothing$. Addition on the hom-classes of $\mathbf{C}^+$ is given by $[f_{i,j}] + [g_{i,j}] = [f_{i,j} + g_{i,j}]$. We now justify the the word "completion" by showing that the natural additive fully faithful functor

$$\begin{aligned} F\colon \mathbf{C} &\to \mathbf{C}^+, \\ a &\mapsto (a) && \text{for all } a \in \mathbf{C}, \\ f &\mapsto [f] && \text{for all } f \in \operatorname{Mor}\mathbf{C}, \end{aligned}$$

satisfies the property that for all additive functors $G\colon \mathbf{C} \to \mathbf{A}$ where $\mathbf{A}$ is additive, there is a unique (up to natural isomorphism) additive functor $H\colon \mathbf{C}^+ \to \mathbf{A}$ such that $H \circ F = G$.

Existence is fairly obvious, simply fix biproducts in $\mathbf{A}$ and let H be defined by $(a_1, \ldots, a_n) \mapsto G(a_1) \oplus \cdots \oplus G(a_n)$ for all $(a_1, \ldots a_n) \in \mathbf{C}^+$. Now, suppose there is another additive functor $H' \colon \mathbf{C}^+ \to \mathbf{A}$ such that $H' \circ F = G$. Let $[f_{i,j}] \colon (a_1, \ldots, a_n) \to (b_1, \ldots b_m)$ be an arbitrary morphism in $\mathbf{C}^+$ with $n, m > 0$ (when $n = 0$ or $m = 0$, the morphism is a zero morphism and the verification is quite straightforward). Define two matrices

$$C_i = \begin{pmatrix} 0_{b_i,b_1} \\ \vdots \\ 0_{b_i,b_{i-1}} \\ \mathrm{id}_{b_i} \\ 0_{b_i,b_{i+1}} \\ \vdots \\ 0_{b_i,b_m} \end{pmatrix} \quad \text{and} \quad R_j = \begin{pmatrix} 0_{a_1,a_j} & \cdots & 0_{a_{j-1},a_j} & \mathrm{id}_{a_j} & 0_{a_{j+1},a_j} & \cdots & 0_{a_n,a_j} \end{pmatrix}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Note that

$$[f_{i,j}] = \sum_{i=1}^{m} \sum_{j=1}^{n} C_i \left(f_{i,j} \right) R_j.$$

Also, note that the matrices

$$C'_i = \begin{pmatrix} 0_{b_1,b_i} & \cdots & 0_{b_{i-1},b_i} & \mathrm{id}_{b_i} & 0_{b_{i+1},b_i} & \cdots 0_{b_m,b_i} \end{pmatrix} \quad \text{and} \quad R'_j = \begin{pmatrix} 0_{a_j,a_1} \\ \cdots \\ 0_{a_j,a_{j-1}} \\ \mathrm{id}_{a_j} \\ 0_{a_j,a_{j+1}} \\ \cdots \\ 0_{a_j,a_n} \end{pmatrix}$$

are respectively left inverses and right inverses for C_i and R_j for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The natural isomorphism between H and H' can now be summarized in the following commutative diagram:

$$\begin{tikzcd}
H(a_1, \ldots, a_n) \arrow[r, "H({[f_{i.j}]})"] \arrow[d, "\alpha"'] & H(b_1, \ldots, b_m) \arrow[d, "\beta"'] \\
H'(a_1, \ldots, a_n) \arrow[r, "H'({[f_{i.j}]})"] & H'(b_1, \ldots, b_m)
\end{tikzcd}$$

where $\alpha = \sum_{j=1}^{n} H'(R'_j)H(R_j)$ and $\beta = \sum_{i=1}^{m} H'(C_i)H(C'_i)$. Note that α and β are isomorphisms since $\alpha^{-1} = \sum_{j=1}^{n} H(R'_j)H'(R_j)$ and $\beta^{-1} = \sum_{i=1}^{m} H(C_i)H'(C'_i)$.

Remark 2.2.26. If $\mathbf{A}$ is an additive category and $\mathbf{A}^+$ is its additive completion, then, as a direct consequence of the universal property, $\mathbf{A}$ and $\mathbf{A}^+$ are equivalent categories.

Alternately, we may prove this assertion by noting that the additive functor $F\colon \mathbf{A} \to \mathbf{A}^+$ given by $a \mapsto (a)$ and $f \mapsto [f]$ is fully faithful. Then, F is essentially surjective because

$$(a_1, \ldots, a_n) \cong (a_1) \oplus \cdots \oplus (a_n) \cong F(a_1) \oplus \cdots \oplus F(a_n) \cong F(a_1 \oplus \cdots \oplus a_n).$$

The last isomorphism above is by Lemma 2.2.20 as the functor is between two additive categories.

2.3 The Krull-Schmidt Property

In this section, we give a more precise description of the split Grothendieck group of categories that have a property called the *Krull-Schmidt property*. This property is useful because if a category has the Krull-Schmidt property, then the split Grothendieck group of the category admits a nice basis. We begin with a motivating example—the category of finitely generated abelian groups.

Example 2.3.1. The split Grothendieck group of $\mathbf{Ab}^{\mathrm{fg}}$ is isomorphic (as a group) to $\mathbb{Z}[x]$. By the fundamental theorem of finitely generated abelian groups, we have that every finitely generated abelian group is isomorphic to a group of the form $\mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_m}$ where $q_1, \ldots, q_m$ are not necessarily distinct powers of primes and $n, m \in \mathbb{N}$. Furthermore, this decomposition is unique up to permutation of the indices q_i. Let $p_1, p_2, \ldots$ denote the primes in increasing order, from which it follows that $\{[\mathbb{Z}]\} \cup \{[\mathbb{Z}_{p_i^j}] \mid i, j \in \mathbb{N}_+\}$ freely generates $G_0(\mathbf{Ab}^{\mathrm{fg}}, \oplus)$. In other words, every element of the split Grothendieck group can be written uniquely in the form

$$g = n_{0,0}[\mathbb{Z}] + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} n_{i,j}[\mathbb{Z}q_i^j],$$

where all but finitely many integers $n_{i,j}$ are zero. One can verify that the map

$$\begin{aligned} G_0(\mathbf{Ab}^{\mathrm{fg}}, \oplus) &\to \mathbb{Z}[x] \\ g &\mapsto n_{0,0} + n_{1,1}x + n_{1,2}x^2 + n_{2,1}x^3 + n_{3,1}x^4 + n_{2,2}x^5 + n_{1,3}x^6 + n_{1,4}x^7 + \cdots . \end{aligned}$$

is a group isomorphism.

Definition 2.3.2 (Indecomposable object)**.** Let $\mathbf{C}$ be a category with an initial object c. An object a is *indecomposable* if whenever $a \cong \coprod_{i \in I} b_i$ where I is some index set, then there is a unique index $j \in I$ such that $a \cong b_j$ and $b_i \cong c$ for all $i \neq j$.

Lemma 2.3.3. *An initial object is not indecomposable.*

Proof. Let $c \in \mathbf{C}$ be initial. By Remark 1.3.8, we have $c \cong c \coprod c$. Hence, the object c cannot satisfy the uniqueness of the index j in Definition 2.3.2 since c is isomorphic to both the first and second instance of c. ☐

Example 2.3.4. In $\mathbf{Ab}^{\mathrm{fg}}$, any group of the form $\mathbb{Z}_q$ where q is a power of a prime is indecomposable. The trivial group $\{\epsilon\}$ is not indecomposable because it is an initial object.

Definition 2.3.5 (Krull-Schmidt property)**.** An additive category $\mathbf{C}$ has the *Krull-Schmidt property* if the following hold:

(a) Every object in $\mathbf{C}$ is isomorphic to $a_1 \oplus \cdots \oplus a_n$ for some $n \in \mathbb{N}$ and indecomposable objects $a_1, \ldots, a_n \in \mathbf{C}$. By convention, the zero object is isomorphic to the case of $n = 0$.

(b) For any $m \in \mathbb{N}$ and indecomposable objects $b_1, \ldots, b_m$ we have

$$\bigoplus_{i=1}^{n} a_i \cong \bigoplus_{i=1}^{m} b_i \implies n = m \text{ and } a_i \cong b_{\rho(i)} \text{ for all } i,$$

where ρ is some permutation of the indices.

Theorem 2.3.6. *Let $\mathbf{A}$ be an essentially small additive category in which the Krull-Schmidt property holds and let S denote the set of equivalence classes of indecomposable objects in $\mathbf{A}$. Then, S is a basis of the split Grothendieck group $G_0(\mathbf{A}, \oplus)$.*

Proof. Let $X = \sum_{i=1}^{n} \alpha_i [x_i]$ be an arbitrary element of $G_0(\mathbf{A}, \oplus)$. By the Krull-Schmidt property, we have

$$[x_i] = \left[\bigoplus_{j=1}^{m_i} a_{i,j} \right] = \sum_{j=1}^{m_i} [a_{i,j}],$$

where $a_{i,j}$ are indecomposable objects for all i and j. Since

$$X = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m_i} [a_{i,j}],$$

we have that $X \in \langle S \rangle$. Since $\alpha[a] = \sum_{i=1}^{\alpha} [a]$ when $\alpha > 0$, let $A = \sum_{i=1}^{n} [a_i] - \sum_{i=1}^{n'} [a'_i]$ and $B = \sum_{i=1}^{m} [b_i] - \sum_{i=1}^{m'} [b'_i]$ be arbitrary elements of $G_0(\mathbf{A}, \oplus)$ where $n, n', m, m' \in \mathbb{N}$ and a_i, a'_i, b_i, and b'_i are indecompsable for all applicable i. Without loss of generality, we do not have $a_i \cong a'_j$ or $b_i \cong b'_j$ for any applicable i or j (since summands that appear in both sums can be cancelled out). Then,

$$A = B \implies \left[\left(\bigoplus_{i=1}^{n} a_i \right) \oplus \left(\bigoplus_{i=1}^{m'} b'_i \right) \right] = \left[\left(\bigoplus_{i=1}^{n'} a'_i \right) \oplus \left(\bigoplus_{i=1}^{m} b_i \right) \right]$$

Hence, by Lemma 2.1.16(b), we have

$$\left(\bigoplus_{i=1}^{n} a_i \right) \oplus \left(\bigoplus_{i=1}^{m'} b'_i \right) \oplus c \cong \left(\bigoplus_{i=1}^{n'} a'_i \right) \oplus \left(\bigoplus_{i=1}^{m} b_i \right) \oplus c$$

for some $c \in \operatorname{Ob} \mathbf{A}$. By the Krull-Shmidt property, $c \cong \bigoplus_{i=1}^{r} c_i$ where $r \in \mathbb{N}$ and c_i is indecomposable for all i. So,

$$\left(\bigoplus_{i=1}^{n} a_i \right) \oplus \left(\bigoplus_{i=1}^{m'} b'_i \right) \oplus \left(\bigoplus_{i=1}^{r} c_i \right) \cong \left(\bigoplus_{i=1}^{n'} a'_i \right) \oplus \left(\bigoplus_{i=1}^{m} b_i \right) \oplus \left(\bigoplus_{i=1}^{r} c_i \right)$$

We apply the Krull-Shmidt property again and thus, $n + m' = n' + m$. Furthermore, each indecomposable object on one side is isomorphic to one on the other. Clearly, $c_i \cong c_i$ for $i = 1, \dots, r$. Since we declared that $a_i \cong a'_j$ for no applicable i and j, we must have $a_i \cong b_j$ for some i and j. We conclude that A and B only differ by the order of the summands. Hence, S is a basis for $G_0(\mathbf{A}, \oplus)$. □

Definition 2.3.7 (Endomorphism, idempotent)**.** A morphism f is an *endomorphism* if $\mathrm{dom}(f) = \mathrm{cod}(f)$. An endomorphism is an *idempotent* if $f \circ f = f$.

Definition 2.3.8 (Invariant basis number (IBN), IBN category)**.** A ring R satisfies the *invariant basis number* (IBN) property if whenever $R^n \cong R^m$ in R-**Mod** with $n, m \in \mathbb{N}$, then $n = m$.

Let $\mathcal{F}_R$ denote the full subcategory of R-**Mod** consisting of finitely generated free left R-modules. An *IBN category* is an essentially small additive category where all idempotents have a kernel and, in addition, is equivalent to $\mathcal{F}_R$ for some ring R that satisfies the (IBN) property.

Example 2.3.9. Any non-trivial commutative ring satisfies the IBN property. For any non-trivial field $\mathbb{K}$, the category $\mathbf{FinVect}_{\mathbb{K}}$ is clearly an IBN category.

Definition 2.3.10 (Coproduct category)**.** Let $\{\mathbf{C}_j \mid j \in J\}$ be a collection of categories. We define the *coproduct category* of these categories, denoted $\coprod_{j \in J} \mathbf{C}_j$, as follows:

$$\mathrm{Ob} \coprod_{j \in J} \mathbf{C}_j = \{(j, c) \mid j \in J \text{ and } c \in \mathbf{C}_j\}$$

$$\mathrm{Hom}_{\coprod_{j \in J} \mathbf{C}_j}((j, c), (j', c')) = \begin{cases} \mathrm{Hom}_{\mathbf{C}_j}(c, c') & \text{if } j = j' \\ \varnothing & \text{if } j \neq j' \end{cases}$$

where composition is given by composition on $\mathbf{C}_j$ for all j.

Theorem 2.3.11. *Let $\mathbf{C}$ be an essentially small additive category where all idempotents have a kernel. The following are equivalent:*

(a) There is some family $\{\mathbf{C}_i \mid i \in I\}$ of IBN categories and an additive functor $F\colon \mathbf{C} \to \coprod_{i \in I} \mathbf{C}_i$ that is essentially surjective and isomorphism reflecting.

(b) The split Grothendieck group $G_0(\mathbf{C}, \oplus)$ is free.

(c) The category $\mathbf{C}$ has the Krull-Schmidt property.

Proof. The details of the proof are beyond the scope of this paper. We refer the interested reader to [7, p. 127]. □

2.4 Abelian Categories

In this section, we examine Grothendieck groups of abelian categories, which are additive categories with some extra structure, and show that the Grothendieck group of an abelian category is a quotient of the split Grothendieck group (of the same category regarded as an additive category).

Definition 2.4.1 (Abelian category)**.** A category $\mathbf{A}$ is *abelian* if the following hold:

(AB1) The category $\mathbf{A}$ is additive.

(AB2) Every morphism in $\operatorname{Mor}\mathbf{A}$ has both a kernel and a cokernel.

(AB3) Every monomorphism in $\operatorname{Mor}\mathbf{A}$ is a kernel and every epimorphism in $\operatorname{Mor}\mathbf{A}$ is a cokernel.

Proposition 2.4.2. *Abelian categories are balanced.*

Proof. Let $h\colon a \to b$ be a bimorphism. By Lemma 1.2.19, $\ker(h) = 0_{0,a}$. Note also that $\operatorname{coker}(0_{0,a}) = h$. Now, $\mathrm{id}_a \circ 0_{0,a} = 0_{0,a}$, so there exists a unique morphism g satisfying $g \circ h = \mathrm{id}_a$. In addition, $(h \circ g) \circ h = h$ and, since h is a bimorphism, we have $h \circ g = \mathrm{id}_b$. $\square$

Recall that a *sequence* is a set with a bijection to a countable totally ordered set called the *index set*. In some areas of mathematics, such as analysis, it is common to restrict the choice of index sets to subsets of $\mathbb{N}$. For us, it will be more convenient to instead use subsets of $\mathbb{Z}$, as this better suits the diagrams that follow.

Definition 2.4.3 (Exact sequence, short exact sequence)**.** Let $\mathbf{A}$ be an abelian category. A *exact sequence* is a sequence of arrows

$$\cdots \xrightarrow{f_{i-1}} a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_{i+1}} a_{i+2} \xrightarrow{f_{i+2}} \cdots \tag{2.4.1}$$

such that $\ker(f_{i+1}) = f_i$. A *short exact sequence* is an exact sequence of the form

$$\cdots \to 0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0 \to \cdots .$$

We will write simply,

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0.$$

Remark 2.4.4. In a short exact sequence $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$, the morphism f is a monomorphism and the morphism g is an epimorphism. This fact follows from the definition of kernel and the fact that the unnamed morphisms are zero morphisms.

Remark 2.4.5. In $\mathbf{Grp}$, Definition 2.4.3 implies that an exact sequence is a sequence of arrows (2.4.1) such that $\ker(f_{i+1}) = \operatorname{im}(f_i)$ where ker and im are the traditional kernel and image. By Proposition 1.2.29, the kernel of a morphism f can be identified with the inclusion map from the traditional kernel to the domain of f. Hence, $\operatorname{im}(i) = \ker(f)$ where ker and im are the tradition kernel and image. The same is true for many other categories including $\mathbf{Ab}$, $\mathbf{FinAb}$, and $\mathbf{Vect}_{\mathbb{K}}$.

Lemma 2.4.6. *Let* $\mathbf{A}$ *be an abelian category. For any morphisms* $g\colon b \to c$ *and* $g'\colon b' \to c'$*, we have* $\ker(g \oplus g') \cong \ker(g) \oplus \ker(g')$.

Proof. The statement follows from Theorem 3.3.1 and the equivalent statement for modules. The following is an alternate proof. Let $\ker(g) = f$ and $\ker(g') = f'$ and let $\mathrm{dom}(f) = a$ and $\mathrm{dom}(f') = a'$. Fix the obvious notation for biproducts $a \oplus a'$, $b \oplus b'$ and $c \oplus c'$. Since $\oplus$ is a binary operation on $\mathbf{A}$, we have, by Lemma 2.1.2(b),

$$(g \oplus g') \circ (f \oplus f') = (g \circ f) \oplus (g' \circ f') = 0_{a,c} \oplus 0_{a',c'} = 0_{a \oplus a', c \oplus c'}$$

Suppose for some morphism $k\colon d \to b \oplus b'$, we have $(g \oplus g') \circ k = 0$. Applying p_c and $p_{c'}$ to this equality, we get $g \circ (p_b \circ k) = 0$ and $g' \circ (p_{b'} \circ k) = 0$ respectively. By definition of the kernel of g, there are morphisms $u_a\colon d \to a$ and $u_{a'}\colon d \to a'$ such that $f \circ u_a = p_b \circ k$ and $f' \circ u_{a'} = p_{b'} \circ k$. Thus,

$$(f \oplus f') \circ (i_a \circ u_a + i_{a'} \circ u_{a'}) = i_b \circ f \circ u_a + i_{b'} \circ f' \circ u_{a'} = k.$$

Suppose for some w we have $(f \oplus f') \circ w = k$. Since $g \circ (f \circ p_a \circ w) = 0$, we have $p_a \circ w = u_a$ by uniqueness of the factoring morphism in the definition of kernel of g. Similarly, $p_{a'} \circ w = u_{a'}$. Hence, $w = i_a \circ u_a + i_{a'} \circ u_{a'}$. The statement follows since kernels are unique up to isomorphism. □

Corollary 2.4.7. *If*

$$\cdots \xrightarrow{f_{i-1}} a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_{i+1}} a_{i+2} \xrightarrow{f_{i+2}} \cdots$$

and

$$\cdots \xrightarrow{g_{i-1}} b_i \xrightarrow{g_i} b_{i+1} \xrightarrow{g_{i+1}} b_{i+2} \xrightarrow{g_{i+2}} \cdots$$

are exact sequences, then their biproduct sequence

$$\cdots \xrightarrow{f_{i-1} \oplus g_{i-1}} a_i \oplus b_i \xrightarrow{f_i \oplus g_i} a_{i+1} \oplus b_{i+1} \xrightarrow{f_{i+1} \oplus g_{i+1}} a_{i+2} \oplus b_{i+2} \xrightarrow{f_{i+2} \oplus g_{i+2}} \cdots$$

is an exact sequence.

Example 2.4.8. Fix a vector space $\mathbb{K}$ and consider $\mathbf{FinVect}_{\mathbb{K}}$. Recall that the *Rank-Nullity Theorem* states that for any $\mathbb{K}$-linear map $T\colon V \to W$, we have that $\dim_{\mathbb{K}} V = \dim_{\mathbb{K}}(\operatorname{im} T) + \dim_{\mathbb{K}}(\ker T)$. It follows that any short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $\mathbf{FinVect}_{\mathbb{K}}$ satisfies the property that $\dim_{\mathbb{K}} V_2 = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_3$.

Example 2.4.9. In $\mathbf{Ab}$, if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence, then $C \cong B/A$ (identifying A with $\operatorname{im} f$ since f is injective). Denoting the order of a group G by $\mathrm{o}(G)$, we have in $\mathbf{FinAb}$ as a consequence of the First Isomorphism Theorem that $\mathrm{o}(B) = \mathrm{o}(A) \cdot \mathrm{o}(C)$.

Definition 2.4.10 (Additive function)**.** A map ψ from $\operatorname{Ob}\mathbf{C}$ into an abelian group is called *additive* if for every exact sequence

$$0 \to a \to b \to c \to 0$$

we have that $\psi(b) = \psi(a) + \psi(c)$.

Definition 2.4.11. The *rank* of a finitely generated abelian group G, denoted $\operatorname{Rank}(G)$, is the maximum cardinality of a linearly independent set of elements over $\mathbb{Z}$. It follows that every element of such a set must be free in the group, and that every free element in the group must be a linear combination of elements in this set. If $\operatorname{Ob}\mathbf{C}$ consists of abelian groups, then we will also use Rank to denote the following group homomorphism defined by how it acts on its generators:

$$\begin{aligned} G^{\text{iso}}(\mathbf{C}) \to& \mathbb{Z} \\ [G] \mapsto& \operatorname{Rank}(G). \end{aligned}$$

Since any two isomorphic abelian groups share the same rank, this group homomorphism is well defined.

Lemma 2.4.12. *The map* Rank *is additive on* $\mathbf{Ab}^{\text{fg}}$.

Proof. For this example, let all tensor products be over $\mathbb{Z}$. For any $G \in \mathbf{Ab}^{\text{fg}}$, the tensor product $\mathbb{Q}\otimes G$ is a $\mathbb{Q}$ module where $r(q\otimes g) = (rq\otimes g)$ where $r, q \in \mathbb{Q}$ and $g \in G$. If $f : G \to H$ is a group homomorphism, then $f : \mathbb{Q} \otimes G \to \mathbb{Q} \otimes H$ is a $\mathbb{Q}$-module homomorphism given by $f(q \otimes g) = q \otimes f(g)$ and extended by linearity. Now, if $0 \to G \to H \to K \to 0$ is a short exact sequence in $\mathbf{Ab}^{\text{fg}}$, then $0 \to \mathbb{Q} \otimes G \to \mathbb{Q} \otimes H \to \mathbb{Q} \otimes K \to 0$ is a short exact sequence in $\mathbb{Q}$-**Mod**. To show this, note that any tensor of the form $\mathbb{Q} \otimes G$ where G is a group is rank one since $\dim_{\mathbb{Q}} \mathbb{Q} = 1$. Next, let $f : H \to K$ be a group homomorphism and let $f' : \mathbb{Q} \otimes H \to \mathbb{Q} \otimes K$ be its corresponding $\mathbb{Q}$-module homomorphism. Note that

$$q \otimes g \in \ker(f') \iff q = 0 \text{ or } nf(g) = 0 \text{ for some } n \in \mathbb{N}.$$

If $q = 0$, then $q\otimes g = 0 = q\otimes 0 \in \mathbb{Q}\otimes\ker(f)$. If $nf(g) = 0$, then $q\otimes g = \frac{q}{n}\otimes ng \in \mathbb{Q}\otimes\ker(f)$. The converse that $q \otimes g \in \mathbb{Q} \otimes \ker(f)$ means that $q \otimes g \in \ker(f')$ is obvious. Hence, $\ker(f') = \mathbb{Q} \otimes \ker(f)$. Note that $\operatorname{Rank}(G) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes G)$. Since $\mathbb{Q}$ is a field, the Rank-Nullity Theorem applies. $\square$

Definition 2.4.13 (Grothendieck group of an abelian category)**.** Let $\mathbf{A}$ be an essentially small abelian category. Define $N_0(\mathbf{A}) \trianglelefteq G^{\text{iso}}(\mathbf{A})$ to be the (normal) subgroup generated by $\{[b] - [a] - [c] \mid 0 \to a \to b \to c \to 0 \text{ is a short exact sequence}\}$. The *Grothendieck group* $G_0(\mathbf{A})$ of $\mathbf{A}$ is the quotient group $G^{\text{iso}}(\mathbf{A})/N_0(\mathbf{A})$. By a common abuse of terminology, we will write $[a]$ to mean the image of the isomorphism class of $[a]$ in $G_0(\mathbf{A})$ when this will not cause confusion. It follows that for every short exact sequence $0 \to a \to b \to c \to 0$, we have the relation $[b] - [a] - [c] = 0$ (or equivalently, $[b] = [a] + [c]$) in $G_0(\mathbf{A})$.

We will henceforth assume all categories discussed are essentially small, unless otherwise specified.

Proposition 2.4.14. *The Grothendieck group $G_0(\mathbf{A})$ of an abelian category $\mathbf{A}$ satisfies the following universal property. Let $\varphi\colon \operatorname{Ob}\mathbf{A} \to G_0(\mathbf{A})$ be the natural map $a \mapsto [a]$ for all $a \in \operatorname{Ob}\mathbf{A}$. For every additive function $\psi\colon \operatorname{Ob}\mathbf{A} \to H$ where H is an abelian group there is a unique group homomorphism $\theta\colon G_0(\mathbf{A}) \to H$ such that $\theta \circ \varphi = \psi$. This statement can be summarized by the following commutative diagram:*

$$\begin{tikzcd}
\operatorname{Ob}\mathbf{A} \arrow[r, "\psi"] \arrow[d, "\varphi"'] & H \\
G_0(\mathbf{A}) \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

Proof. The morphism $\theta\colon G_0(\mathbf{A}) \to H$ given by $\theta([a]) = \psi(a)$ and extended by linearity is well defined on $G^{\text{iso}}(\mathbf{A})$ because if $a \cong b$ then

$$0 \to 0 \to a \to b \to 0$$

is an exact sequence so $\psi(a) = \psi(b)$. Clearly, for any short exact sequence $0 \to a \to b \to c \to 0$ we have $\theta([b] - [a] - [c]) = \psi(b) - \psi(a) - \psi(c) = 0$ and so θ is well defined on $G_0(\mathbf{A})$. Note that $\theta \circ \varphi = \psi$. Now, if θ' is a group homomorphism that also satisfies $\theta' \circ \varphi = \psi$ we see that

$$\theta'([a]) = (\theta' \circ \varphi)(a) = \psi(a) = (\theta \circ \varphi)(a) = \theta(a)$$

and, thus, the group homomorphism is unique. $\square$

Proposition 2.4.15. *Let $\mathbf{A}$ be an abelian category. Let $a, b \in \operatorname{Ob}\mathbf{A}$. There are morphisms i_a and p_b such that*

$$0 \to a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} b \to 0 \tag{2.4.2}$$

is an exact sequence.

Proof. Let $\mathbf{A}$ be an abelian category. Let $a, b \in \mathbf{A}$. We aim to show that $\ker(p_b) \cong i_a$ where p_b and i_a are the morphisms in the definition of biproduct. By definition, $p_b \circ i_a = 0_{ab}$. Now, suppose there is some $k\colon c \to a \oplus b$ such that $p_b \circ k = 0_{cb}$. Note that the following diagram commutes:

$$\begin{tikzcd}
 & c \arrow[dl, "p_a \circ k"'] \arrow[d, "k"] \arrow[dr, "0_{cb}"] & \\
a & a \oplus b \arrow[l, "p_a"'] \arrow[r, "p_b"] & b
\end{tikzcd}$$

Since $i_a \circ p_a \circ k$ also makes this diagram commute in place of k, we have $k = i_a \circ (p_a \circ k)$. Suppose for some $h\colon c \to a$ we also have $k = i_a \circ h$. Then, $h = p_a \circ i_a \circ h = p_a \circ k$ so the morphism is unique. Hence, $\ker(p_b) \cong i_a$. We leave it to the reader to verify that $\ker(i_a) \cong 0_{0,a}$ and $\ker(0_{c,0}) \cong p_b$. Therefore, (2.4.2) is an exact sequence. $\square$

Corollary 2.4.16. *Let* $\mathbf{A}$ *be an abelian category. The Grothendieck group* $G_0(\mathbf{A})$ *is a quotient of the split Grothendieck group* $G_0(\mathbf{A}, \oplus)$.

Lemma 2.4.17. *Let* $\mathbf{A}$ *be an essentially small abelian category and let* 0 *be its zero object. Then,* $[0]$ *is the identity element of* $G_0(\mathbf{A})$.

Proof. For every $C \in \mathbf{C}$ we have the short exact sequence:

$$0 \to C \overset{\mathrm{id}_C}{\to} C \to 0 \to 0$$

Hence, $[C] = [C] + [0]$. Since $G_0(\mathbf{A})$ is generated by the elements $[C]$ where $C \in \mathbf{A}$, all elements of $G_0(\mathbf{A})$ are of the form $m_1[C_1] + \ldots + m_n[C_n]$ for some integers $m_1, \ldots, m_n$ and some (not necessarily distinct) objects $C_1, \ldots, C_n$ of $\mathbf{A}$. We thus have for all elements of $G_0(\mathbf{A})$:

$$\begin{aligned}[0] + m_1[C_1] + \ldots + m_n[C_n] &= [0] + [C_1] + (m_1 - 1)[C_1] + \ldots + m_n[C_n] \\ &= m_1[C_1] + \ldots + m_n[C_n]\end{aligned}$$

□

Lemma 2.4.18. *Let* $\mathbf{A}$ *be an abelian category. Then, for every short exact sequence* $0 \to A \to B \to C \to 0$ *and any object* $D \in \mathbf{A}$*, there exist short exact sequences*

$$0 \to A \oplus D \to B \oplus D \to C \to 0$$

and

$$0 \to A \to B \oplus D \to C \oplus D \to 0.$$

Proof. We apply Corollary 2.4.7 to $0 \to A \to B \to C \to 0$ together with $0 \to D \overset{\mathrm{id}}{\to} D \to 0 \to 0$ and $0 \to 0 \to D \overset{\mathrm{id}}{\to} D \to 0$, respectively. □

Proposition 2.4.19. *Let* $\mathbf{A}$ *be an abelian category with* $A, B \in \mathbf{A}$*. Then, the following are equivalent:*

(a) $[A] = [B]$ *in* $G_0(\mathbf{A})$.

(b) There exist objects C, U, V *such that*

$$0 \to U \to A \oplus C \to V \to 0$$

and

$$0 \to U \to B \oplus C \to V \to 0$$

are short exact sequences.

Proof. Assume first that (b) holds. Then, we have

$$[A] + [C] = [A \oplus C] = [U] + [V] = [B \oplus C] = [B] + [C],$$

and thus, $[A] = [B]$.

Now assume that (a) holds. Then, $[A]-[B] \in N_0(\mathbf{A})$, so there exist nonnegative integers n, m and exact sequences

$$0 \to X_i \to Y_i \to Z_i \to 0,$$
$$0 \to X'_j \to Y'_j \to Z'_j \to 0,$$

for all $0 \leq i \leq n$ and all $0 \leq j \leq m$ giving us the following equation in $G^{\mathrm{iso}}(\mathbf{A})$:

$$[A]-[B] = \sum_{i=0}^{n}([Y_i]-[X_i]-[Z_i]) - \sum_{j=0}^{m}([Y'_j]-[X'_j]-[Z'_j]).$$

Rearranging terms to eliminate negative expressions, we obtain

$$[A] + \sum_{i=0}^{n}[Y_i] + \sum_{j=0}^{m}([X'_j]+[Z'_j]) = [B] + \sum_{i=0}^{n}[Y'_i] + \sum_{j=0}^{m}([X_j]+[Z_j]) \tag{2.4.3}$$

Now define

$$X = \bigoplus_{i=0}^{n} X_i, \qquad Y = \bigoplus_{i=0}^{n} Y_i, \qquad Z = \bigoplus_{i=0}^{n} Z_i,$$
$$X' = \bigoplus_{j=0}^{m} X'_i, \qquad Y' = \bigoplus_{j=0}^{m} Y'_i, \qquad Z' = \bigoplus_{j=0}^{m} Z'_i.$$

By similar reasoning as in Lemma 2.1.16(b), we conclude from (2.4.3) that

$$A \oplus Y \oplus X' \oplus Z' \cong B \oplus Y' \oplus X \oplus Z. \tag{2.4.4}$$

By Corollary 2.4.7, we have short exact sequences:

$$0 \to X \to Y \to Z \to 0 \tag{2.4.5}$$

and

$$0 \to X' \to Y' \to Z' \to 0. \tag{2.4.6}$$

Applying Lemma 2.4.18 to (2.4.5), we obtain a short exact sequence

$$0 \to X \oplus X' \to Y \oplus X' \to Z \to 0,$$

and applying Lemma 2.4.18 again we obtain another short exact sequence

$$0 \to X \oplus X' \to (Y \oplus X') \oplus (A \oplus B \oplus Z') \to Z \oplus (A \oplus B \oplus Z') \to 0,$$

which we rearrange to become

$$0 \to X \oplus X' \to B \oplus (A \oplus Y \oplus X' \oplus Z') \to B \oplus A \oplus Z \oplus Z' \to 0.$$

Now setting $U = X \oplus X'$, $C = A \oplus Y \oplus X' \oplus Z'$, $V = B \oplus A \oplus Z \oplus Z'$, the last sequence above becomes

$$0 \to U \to B \oplus C \to V \to 0.$$

Similarly, applying Lemma 2.4.18 to (2.4.6) together with the isomorphism (2.4.4), we obtain a short exact sequence

$$0 \to U \to A \oplus C \to V \to 0. \qquad \square$$

Example 2.4.20. The Grothendieck group $G_0(\mathbf{Ab}^{\mathrm{fg}})$ is isomorphic to $\mathbb{Z}$ as a group. By Proposition 2.4.15, we have $[\mathbb{Z}^n] = [\bigoplus_{i=1}^n \mathbb{Z}] = n[\mathbb{Z}]$. For every $n \in \mathbb{N}_+$,

$$0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \xrightarrow{\text{mod } n} \mathbb{Z}_n \to 0.$$

is a short exact sequence. Hence, $[\mathbb{Z}_n] = 0$ for all $n > 0$. We have shown in Lemma 2.4.12 that the group homomorphism

$$G^{\mathrm{iso}}(\mathbf{Ab}^{\mathrm{fg}}) \xrightarrow{\text{Rank}} \mathbb{Z}$$

is additive, so clearly $N_0(\mathbf{Ab}^{\mathrm{fg}}) \subseteq \ker(\text{Rank})$. Suppose $x \in \ker(\text{Rank})$. By what we have noted thus far, $x = \alpha[\mathbb{Z}]$ in $G_0(\mathbf{Ab}^{\mathrm{fg}})$ for some integer α. Since $0 = \text{Rank}(x) = \alpha \,\text{Rank}([\mathbb{Z}]) = \alpha$, we have $x \in N_0(\mathbf{Ab}^{\mathrm{fg}})$. Hence, $\ker(\text{Rank}) = N_0(\mathbf{Ab}^{\mathrm{fg}})$. Furthermore, since Rank is surjective ($n[\mathbb{Z}] \mapsto n$), we have by the First Isomorphism Theorem ([19, p. 135]):

$$G_0(\mathbf{Ab}^{\mathrm{fg}}) = G^{\mathrm{iso}}(\mathbf{Ab}^{\mathrm{fg}})/\ker(\text{Rank}) \cong \mathbb{Z}.$$

Example 2.4.21. The Grothendieck group $G_0(\mathbf{FinAb})$ is isomorphic to $\mathbb{Z}[x]$ as a group. Let p_i denote the primes in increasing order. Consider the group homomorphism (additive with respect to biproduct) defined by its action on generators of $G^{\mathrm{iso}}(\mathbf{FinAb})$:

$$\begin{aligned} \text{Card}\colon G^{\mathrm{iso}}(\mathbf{FinAb}) \to& \mathbb{Z}[x], \\ [G] \mapsto& r_0 + r_1 x + r_2 x^2 + \cdots, \end{aligned}$$

where $|G| = p_0^{r_0} p_1^{r_1} \cdots$. This is well-defined because isomorphic groups share the same order and the natural numbers with multiplication form a free commutative monoid. By Example 2.4.9, we have $\text{Card}([B] - [A] - [C]) = \text{Card}([B]) - \text{Card}([A]) - \text{Card}([B/A]) = 0$. Therefore, we have $N_0(\mathbf{FinAb}) \subseteq \ker \text{Card}$. For the other inclusion, we observe that any nontrivial element in $\mathbf{FinAb}$ can be written uniquely in the form $\bigoplus_{i=1}^r \mathbb{Z}_{q_i}$ where q_i are powers of primes and $r \in \mathbb{N}$ (up to isomorphism and up to permutation of the q_i). So, since $G_0(\mathbf{FinAb})$ is a quotient of $G_0(\mathbf{FinAb}, \oplus)$, the set $\{[\mathbb{Z}_q] \mid q \text{ is a power of a prime}\}$ generates $G_0(\mathbf{FinAb})$. Furthermore, for any prime p and $k \in \mathbb{N}_+$,

$$0 \to \mathbb{Z}_p \xrightarrow{p^{k-1}\cdot} \mathbb{Z}_{p^k} \xrightarrow{\text{mod } p} \mathbb{Z}_{p^{k-1}} \to 0 \tag{2.4.7}$$

is a short exact sequence. Hence, the finite abelian groups of prime order generate $G_0(\mathbf{FinAb})$. Suppose now that $x \in G^{\mathrm{iso}}(\mathbf{FinAb})$ and $\text{Card}(x) = 0$. Then, for some integers α_i, we have $x = \alpha_0[\mathbb{Z}_{p_0}] + \alpha_1[\mathbb{Z}_{p_1}] + \cdots$ in $G_0(\mathbf{FinAb})$ and hence, $\text{Card}(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \cdots = 0 \implies \alpha_0 = \alpha_1 = \cdots = 0 \implies x = 0$. Thus, $\ker(\text{Card}) = N_0(\mathbf{FinAb})$ and so by the First Isomorphism Theorem:

$$G_0(\mathbf{FinAb}) = G^{\mathrm{iso}}(\mathbf{FinAb})/\ker(\text{Card}) \cong \mathbb{Z}[x].$$

Example 2.4.22. Fix a field $\mathbb{K}$. We show that $G_0(\mathbf{FinVect}_{\mathbb{K}}) \cong \mathbb{Z}$. Elements of $G^{\mathrm{iso}}(\mathbf{FinVect}_{\mathbb{K}})$ are of the form $\sum_{i=1}^n a_i[V_i]$ with $a_i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Define a map:

$$\begin{aligned} \nu\colon G^{\mathrm{iso}}(\mathbf{FinVect}_{\mathbb{K}}) &\to \mathbb{Z}, \\ \sum_{i=1}^n a_i[V_i] &\mapsto \sum_{i=1}^n a_i(\dim_{\mathbb{K}} V_i). \end{aligned}$$

As the reader should verify, ν is a group homomorphism. We claim that $\ker(\nu) = N_0(\mathbf{FinVect}_{\mathbb{K}})$.

It follows from Example 2.4.8 that any generator of $N_0(\mathbf{FinVect}_{\mathbb{K}})$ as in Definition 2.4.13 is mapped to 0 by ν. Thus, $N_0(\mathbf{FinVect}_{\mathbb{K}}) \subseteq \ker(\nu)$.

It follows from Proposition 2.4.15 that, in $G_0(\mathbf{FinVect}_{\mathbb{K}})$, $[\mathbb{K}^n] = [\bigoplus_{i=1}^n \mathbb{K}] = n[\mathbb{K}]$ for every $n \in \mathbb{N}_+$, so $G_0(\mathbf{FinVect}_{\mathbb{K}})$ is generated by $[\mathbb{K}]$. Now consider $y \in \ker(\nu)$. We have that $y = m[\mathbb{K}]$ in $G_0(\mathbf{FinVect}_{\mathbb{K}})$ for some $m \in \mathbb{Z}$, so $0 = \nu(y) = m(\nu[\mathbb{K}]) = m$ and so $y \in N_0(\mathbf{FinVect}_{\mathbb{K}})$. Hence, $\ker(\nu) = N_0(\mathbf{FinVect}_{\mathbb{K}})$. It is clear that ν is surjective. Thus, by the First Isomorphism Theorem, $G_0(\mathbf{FinVect}_{\mathbb{K}}) = G^{\text{iso}}(\mathbf{FinVect}_{\mathbb{K}})/N_0(\mathbf{FinVect}_{\mathbb{K}}) \cong \mathbb{Z}$.

2.5 The Jordan-Hölder Theorem

Just as in the case of additive categories with the Krull-Schmidt property, certain abelian categories whose Grothendieck groups admit a nice bases can be similarly described. We will prove the category-theoretic version of the Jordan-Hölder Theorem, which leads to the main result of this section. We begin by continuing where we left off in Section 1.4, with the dual notion of a subobject.

Definition 2.5.1 (Quotient object)**.** Let $\mathbf{C}$ be a category and let $A \in \mathbf{C}$. A *quotient object* of A is a class of equivalent epimorphisms with domain A. If $e\colon A \to B$ is a representative of a quotient object, we will often refer to the object B as a quotient object. If S is a subobject of A, then for a short exact sequence $0 \to S \to A \to C \to 0$, C is a quotient object of A and we write $C = A/S$.

Definition 2.5.2 (Filtration, composition series, composition factor)**.** Let $\mathbf{C}$ be an abelian category with $A \in \mathbf{C}$. A *filtration* for A is a descending chain

$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n = 0$$

such that A_{i+1} is a subobject of A_i for each i. If A_i/A_{i+1} is a simple object for each i, then the filtration is called a *composition series*, the objects A_i/A_{i+1} are called the *composition factors* of the series, and the nonnegative integer n is called the *length* of the composition series.

Definition 2.5.3 (Equivalent composition series)**.** Two composition series of an object are *equivalent* if they have equal length and their composition factors are isomorphic up to permutation.

Example 2.5.4. Consider $\mathbf{Ab}$. Let $G = \mathbb{Z}_{12}$, $H_6 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}$, $H_4 = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$, $H_3 = \{\bar{0}, \bar{4}, \bar{8}\}$, and $H_2 = \{\bar{0}, \bar{6}\}$. Then, G has three equivalent composition series, with the corresponding composition factors in order, as follows:

Composition Series	Composition Factors
$\mathbb{Z}_{12} \supseteq H_6 \supseteq H_3 \supseteq \{\bar{0}\}$	$\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3$
$\mathbb{Z}_{12} \supseteq H_6 \supseteq H_2 \supseteq \{\bar{0}\}$	$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$
$\mathbb{Z}_{12} \supseteq H_4 \supseteq H_2 \supseteq \{\bar{0}\}$	$\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2$

Definition 2.5.5 (Artinian/noetherian category, finite length)**.** An abelian category $\mathbf{C}$ is *artinian* if for all $A \in \mathbf{C}$, every descending chain of subobjects

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

becomes stationary (that is, satisfies the property that, for some $i \in \mathbb{N}$, $A_i \cong A_j$ for all $j \geq i$). Similarily, $\mathbf{C}$ is *noetherian* if for all $A \in \mathbf{C}$, every ascending chain of subobjects

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A$$

becomes stationary. A category which is both artinian and noetherian is said to have *finite length*.

Lemma 2.5.6. *Every nonzero object in a noetherian category has a maximal subobject.*

Proof. Let $\mathbf{C}$ be a noetherian category. Suppose, on the contrary, that a nonzero object $A \in \mathbf{C}$ has no maximal subobject. In particular, A must have a nontrivial subobject A_0, else 0 is a maximal subobject. We form a chain

$$0 \subsetneq A_0 \subsetneq A.$$

Since A_0 is not a maximal subobject of A, there exists an $A_1 \in \mathbf{C}$ such that our chain can be refined to

$$0 \subsetneq A_0 \subsetneq A_1 \subsetneq A.$$

We iterate this process and form an infinite ascending chain that does not become stationary, contradicting the fact that $\mathbf{C}$ is noetherian. Hence, A must have a maximal subobject. □

Proposition 2.5.7. *Let $A, B \in \mathbf{C}$. Then B is a maximal subobject of A if and only if the quotient A/B is a simple object.*

Proof. While the statement of this proposition is a generalization of the analogous statement for modules, the details of the proof are quite involved. We will prove only the reverse implication and refer the reader to references in the literature for justification of certain steps of the argument.

Let $m\colon B \to A$ be a maximal subobject of A. By definition, m is not an isomorphism and the existence of an object B' together with monomorphisms m_1, m_2

$$\begin{tikzcd} B \arrow[r, "m_1"] \arrow[rr, bend left, "m"] & B' \arrow[r, "m_2"] & A \end{tikzcd}$$

sitting in a commutative diagram imply that either m_1 or m_2 is an isomorphism and that the other is thus equivalent to m. We have, by definition, a short exact sequence

$$0 \to B \xrightarrow{m} A \xrightarrow{n} A/B \to 0$$

wherein n is the cokernel of m (and hence n is an epimorphism). Note that since m is not an isomorphism, $A/B \neq 0$. Let $s\colon T \to A/B$ be a monomorphism. By [10, Lemma 2.1.5], pullbacks always exist in abelian categories, so there exists an object P with morphisms $s' \colon P \to A$ and $n' \colon P \to T$ which gives us a pullback square

$$\begin{tikzcd} P \arrow[r, "n'"] \arrow[d, "s'"'] & T \arrow[d, "s"] \\ A \arrow[r, "n"'] & A/B \end{tikzcd}$$

Since s is a monomorphism in an abelian category, it is a kernel; so by Lemma 1.3.12, s' is a kernel and hence also a monomorphism. Since n is an epimorphism, then by [15, p. 203, Proposition 2], our pullback square is also a pushout square and so n' is an epimorphism. By [23, Lemma 3.2], pullbacks preserve kernels, so there is a morphism $m' \colon B \to P$ such that m' is the kernel of n' and which gives us the commutative diagram:

$$\begin{tikzcd} & P \arrow[r, "n'"] \arrow[d, "s'"] & T \arrow[d, "s"] \\ B \arrow[ur, "m'"] \arrow[r, "m"'] & A \arrow[r, "n"'] & A/B \end{tikzcd}$$

Since $m \colon B \to A$ is a maximal subobject, either s' or m' is an isomorphism.

If m' is an isomorphism, then we have that $s \circ n' = n \circ s' = n \circ m \circ (m')^{-1} = 0_{P,A/B}$. This gives us $s \circ n' = 0_{P,A/B} = 0_{T,A/B} \circ n'$ and hence $s = 0_{T,B/A}$ since n' is an epimorphism.

If, on the other hand, s' is an isomorphism, then s' is both a kernel and a cokernel; hence s is both a kernel and a cokernel (as our square is both a pullback and a pushout), and so s is a bimorphism. Since abelian categories are balanced, this gives us that s is an isomorphism.

Therefore, s is either a zero morphism or an isomorphism, so the quotient object A/B is simple. $\square$

Lemma 2.5.8. *Let $\mathbf{C}$ be an abelian category and let $A, B, C \in \mathbf{C}$. If A, B are both subobjects of C, then the follow hold:*

(a) The following diagram is commutative, and every row and column is a short exact sequence:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \cap B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/(A \cap B) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & A \cup B \arrow[r] \arrow[d] & (A \cup B)/A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A/(A \cap B) \arrow[r] \arrow[d] & (A \cup B)/B \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

In particular, we have that $(A \cup B)/B \cong A/(A \cap B)$ and $(A \cup B)/A \cong B/(A \cap B)$.

(b) *If A and B are nonequivalent subobjects and C/A and C/B are simple, then $C = A \cup B$.*

(c) *If $A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n = 0$ is a composition series for A and B is a subobject of A such that A/B is simple, then $A_1 \cap B$ has a composition series.*

Proof. We refer the reader to [20, pp. 172–175] for proof of these assertions. □

Theorem 2.5.9 (Jordan-Hölder Theorem)**.** *Let $\mathbf{C}$ be an abelian category and $A \in \mathbf{C}$. Then, the following hold:*

(a) *If A has a composition series, then any other composition series for A is equivalent.*

(b) *If $\mathbf{C}$ is of finite length, then A has a composition series.*

Proof. The proof is, using the categorical analogues of various module-theoretic definitions and results that we have developed, a generalization of the proof of the Jordan-Hölder Theorem for modules (see, for instance, [22, Theorems 8.15–8.18]). Throughout this proof, we shall use $\sim$ to denote equivalence of composition series.

We prove (a) by induction on the length of composition series. If A has a composition series of length 0 or 1, then A is the zero object or is simple respectively, and the result is clear. Assume now that for some $n \in \mathbb{N}_+$, if an object has a composition series of length $\leq n-1$, then all other composition series for it are equivalent. Now let

$$S_1\colon A = B_0 \supseteq B_1 \supseteq \ldots \supseteq B_n = 0$$

and

$$S_2\colon A = C_0 \supseteq C_1 \supseteq \ldots \supseteq C_m = 0$$

be two composition series for A. If B_1 and C_1 are equivalent subobjects of A, then since the theorem already holds for objects with series of length at most $n-1$ (i.e. for $S_1'\colon B_1 \supseteq \ldots \supseteq B_n = 0$), we have that $S_2'\colon C_1 \supseteq \ldots \supseteq C_m = 0$ is equivalent to S_1' and we are done. Now suppose that B_1 and C_1 are not equivalent. By part (b) of Lemma 2.5.8, $A = B_1 \cup C_1$, and so by part (a) of Lemma 2.5.8, $A/B_1 \cong C_1/(B_1 \cap C_1)$ and $A/C_1 \cong B_1/(B_1 \cap C_1)$. In particular, $C_1/(B_1 \cap C_1)$ and $B_1/(B_1 \cap C_1)$ are simple. By part (c) of Lemma 2.5.8, $B_1 \cap C_1$ has some composition series $L\colon B_1 \cap C_1 \supseteq L_0 \supseteq \ldots \supseteq L_s = 0$. Now consider the following composition series for A:

$$T_1\colon A = B_0 \supseteq B_1 \supseteq B_1 \cap C_1 \supseteq L_0 \supseteq \ldots \supseteq L_s = 0$$

and

$$T_2\colon A = B_0 \supseteq C_1 \supseteq B_1 \cap C_1 \supseteq L_0 \supseteq \ldots \supseteq L_s = 0.$$

We will denote the corresponding composition series for B_1 and C_1 by T_1' and T_2' the same way as we did for S_1 and S_2. We form the diagram:

$$\begin{tikzcd}
S_1\colon & 0 = B_n \arrow[r] & B_{n-1} \arrow[r] & \ldots \arrow[r] & B_2 \arrow[r] & B_1 \arrow[dr] & \\
T_1 \,\&\, T_2\colon & 0 = L_s \arrow[r] & L_{s-1} \arrow[r] & \ldots \arrow[r] & B_1 \cap C_1 \arrow[ur] \arrow[dr] & & B_1 \cup C_1 = A \\
S_2\colon & 0 = C_m \arrow[r] & C_{m-1} \arrow[r] & \ldots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[ur] &
\end{tikzcd}$$

Since the theorem already holds for B_1, we have that $S_1' \sim T_1'$. But then, T_2' also has length $n-1$ and so $S_2' \sim T_2'$ and we have $n = m$. It follows directly that $S_1 \sim T_1$ and $S_2 \sim T_2$. We see that T_1 and T_2 differ only by the two rightmost factors in the above diagram, which we have already noted are isomorphic, hence $T_1 \sim T_2$. Thus, we have $S_1 \sim T_1 \sim T_2 \sim S_2$, so S_1 and S_2 are equivalent. Since A, S_1, and S_2 were arbitrary, the result holds for any object with a composition series of length $\leq n$, which completes the inductive step of the argument.

We now prove (b). Let $\mathbf{C}$ be a category of finite length with $A \in \mathbf{C}$. If A is the zero object or is simple, then A has a composition series of length 0 or 1 respectively, and we are done. Now assume that A is nonzero and not simple. Then, since $\mathbf{C}$ is noetherian, we have by Lemma 2.5.6 that A has a maximal nonzero subobject A_1, and by Proposition 2.5.7 that A/A_1 is a simple object. We form a chain

$$A \supseteq A_1.$$

If A_1 is simple, then we are done. Otherwise, A_1 must have a maximal nonzero subobject A_2 whence A_1/A_2 is again simple, and our chain becomes

$$A \supseteq A_1 \supseteq A_2.$$

If A_2 is simple, then we are done. We continue this way, and since $\mathbf{C}$ is also artinian, this chain must become stationary – that is, for some $i \in \mathbb{N}_+$, we must have that A_i is simple (i.e. the maximal subobject of A_i is 0). Thus, all objects in an abelian category of finite length must have a composition series, which together with part (a) of the theorem, is unique up to equivalence. □

Theorem 2.5.10. *Let $\mathbf{C}$ be an abelian category of finite length. Then,*

$$S = \{[S_i] \mid S_i \text{ is a simple object in } \mathbf{C}\}$$

is a basis for $G_0(\mathbf{C})$.

Proof. The proof is a generalization of the case of categories of modules that satisfy the Jordan-Hölder property (see Theorem 3.3.1). We will give a proof that S generates $G_0(\mathbf{C})$ and refer readers interested in the linear independence of S to [22, Theorem 7.87] for details in the case of categories of modules.

Let A be an arbitrary object in $\mathbf{C}$. Let $S\colon A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = 0$ be a composition series for A, whose existence is guaranteed by and whose length is uniquely determined by the Jordan-Hölder Theorem. If S has length 0, then A is the zero object and there is nothing to prove.

Now assume S has length $n \geq 1$ and write $H_i = A_i/A_{i+1}$ for all $0 \leq i \leq n-1$. We claim that $[A_{n-i}] = [H_{n-i}] + \cdots + [H_{n-1}]$ for all $1 \leq i \leq n$. We will prove this claim by induction on i.

The case of $i = 1$ is clear; $[A_{n-1}] = [A_{n-1}/0] = [A_{n-1}/A_n] = [H_{n-1}]$. Suppose now that the result holds for some $1 \leq j \leq n-1$. We have the canonical short exact sequence $0 \to A_{n-j} \to A_{n-(j+1)} \to H_{n-(j+1)} \to 0$, and hence $[A_{n-(j+1)}] = [H_{n-(j+1)}] + [A_{n-j}] = [H_{n-(j+1)}] + [H_{n-j}] + \cdots + [H_{n-1}]$. Thus, $[A] = [A_0] = \sum_{i=0}^{n-1} [H_i]$ and so S generates $G_0(\mathbf{C})$. □

2.6 Triangulated Categories

In this section, we provide the definition and some immediate properties of a *triangulated category*. Then, we study some properties of the corresponding Grothendieck group. We also briefly give the reader a motivating example of the *homotopy category* of an additive category.

Definition 2.6.1 (Triangle)**.** Let $\mathbf{A}$ be an additive category and let $T\colon \mathbf{A} \to \mathbf{A}$ be an additive automorphism (an *automorphism* is an isomorphism that is an endomorphism). A *triangle* is a sequence of morphisms of the form

$$a \to b \to c \to T(a).$$

Definition 2.6.2 (Triangulated category)**.** A *triangulated category* is a triple $(\mathbf{A}, T, D)$ consisting of an additive category $\mathbf{A}$, an additive automorphism of categories $T\colon \mathbf{A} \to \mathbf{A}$, and a class D consisting of certain triangles, which we call *distinguished triangles*, that satisfy the following axioms:

(TC1) For all objects $a \in \mathbf{A}$, the triangle $a \xrightarrow{\mathrm{id}_a} a \to 0 \to T(a)$ is a distinguished triangle, called the *trivial distinguished triangle* of a.

(TC2) Any morphism $f\colon a \to b$ can be completed to a distinguished triangle $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$.

(TC3) Distinguished triangles are closed under isomorphism. That is, if $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ is a distinguished triangle and $k\colon a \to a'$, $r\colon b \to b'$ and $s\colon c \to c'$ are isomorphisms, then

$$a' \xrightarrow{r \circ f \circ k^{-1}} b' \xrightarrow{s \circ g \circ r^{-1}} c' \xrightarrow{T(k) \circ h \circ s^{-1}} T(a')$$

is also a distinguished triangle.

(TC4) The triangle $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ is a distinguished triangle if and only if $b \xrightarrow{g} c \xrightarrow{h} T(a) \xrightarrow{-T(f)} T(b)$ is a distinguished triangle. Obtaining a triangle of the latter form from the former or one of the former from the latter is known as *shifting* to the right or to the left, respectively.

(TC5) If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ and $a' \xrightarrow{f'} b' \xrightarrow{g'} c' \xrightarrow{h'} T(a')$ are distinguished triangles and there are morphisms $k\colon a \to a'$ and $r\colon b \to b'$ such that $r \circ f = f' \circ k$, then there exists a (not necessarily unique) morphism $u\colon c \to c'$ such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "f"'] \arrow[d, "k"'] & b \arrow[r, "g"'] \arrow[d, "r"'] & c \arrow[r, "h"'] \arrow[d, dotted, "u"'] & T(a) \arrow[d, "T(k)"'] \\
a' \arrow[r, "f'"] & b' \arrow[r, "g'"] & c' \arrow[r, "h'"] & T(a')
\end{tikzcd}$$

(TC6) If $a \xrightarrow{f} b \xrightarrow{g} c' \xrightarrow{h} T(a)$, $b \xrightarrow{j} c \xrightarrow{k} a' \xrightarrow{m} T(b)$, and $a \xrightarrow{j \circ f} c \xrightarrow{n} b' \xrightarrow{q} T(a)$ are distinguished triangles, then there exists a triangle triangle $c' \xrightarrow{u} b' \xrightarrow{v} a' \xrightarrow{w} T(c')$ such that the following diagram commutes:

$$
\begin{tikzcd}
b \arrow[r, "j"'] \arrow[d, "g"] & c \arrow[d, "n"] \arrow[dr, "k"] & \\
c' \arrow[r, "u"'] \arrow[dr, "h"'] & b' \arrow[r, "v"'] \arrow[d, "q"] & a' \arrow[d, "m"] \arrow[dr, "w"] & \\
& T(a) \arrow[r, "T(f)"'] & T(b) \arrow[r, "T(g)"'] & T(c')
\end{tikzcd}
$$

We will often drop T and D from our notation and denote a triangulated category by simply $\mathbf{A}$ when this will not cause confusion.

Lemma 2.6.3. *Let* $\mathbf{A}$ *be a triangulated category and let* $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ *be a distinguished triangle. Then the morphisms* $g \circ f$ *and* $h \circ g$ *are zero morphisms.*

Proof. By (TC4), the triangle $b \xrightarrow{g} c \xrightarrow{h} T(a) \xrightarrow{-T(f)} T(b)$ is a distinguished triangle. By (TC1), $c \xrightarrow{\mathrm{id}_c} c \to 0 \to T(c)$ is a distinguished triangle. By (TC5), there is a morphism u such that the following diagram commutes:

$$
\begin{tikzcd}
b \arrow[r, "g"'] \arrow[d, "g"'] & c \arrow[r, "h"'] \arrow[d, "\mathrm{id}_c"'] & T(a) \arrow[r, "-T(f)"'] \arrow[d, dotted, "u"'] & T(b) \arrow[d, "T(g)"'] \\
c \arrow[r, "\mathrm{id}_c"'] & c \arrow[r] & 0 \arrow[r] & T(c)
\end{tikzcd}
$$

Hence, $0 = T(g) \circ (-T(f)) = -T(g \circ f)$ and since T is an automorphism, $g \circ f = 0$. The fact that $h \circ g = 0$ follows from (TC4) and the same argument as before. $\square$

Lemma 2.6.4. *Let* $\mathbf{A}$ *be a triangulated category and let* $x \in \mathbf{A}$. *If* $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ *is a distinguished triangle, then*

$$\mathrm{Hom}(x,a) \xrightarrow{\mathrm{Hom}(x,f)} \mathrm{Hom}(x,b) \xrightarrow{\mathrm{Hom}(x,g)} \mathrm{Hom}(x,c) \xrightarrow{\mathrm{Hom}(x,h)} \mathrm{Hom}(x,T(a)) \xrightarrow{\mathrm{Hom}(x,T(f))} \mathrm{Hom}(x,T(b))$$

is an exact sequence.

Proof. By (TC4), it suffices to prove that the sequence is exact at $\mathrm{Hom}(x,b)$. Let $k \colon x \to b$ and suppose $\mathrm{Hom}(x,g)(k) = 0_{x,c}$. By definition, this means $g \circ k = 0_{x,c}$. By (TC1) and

(TC4), the rows of the following diagram are distinguished triangles and, by (TC5), there exists a morphism $r\colon x \to a$ that make the diagram commute:

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "k"'] & 0 \arrow[r] \arrow[d] & T(x) \arrow[r, "-T(\mathrm{id}_x)"'] \arrow[d, dotted, "T(r)"'] & T(x) \arrow[d, "T(k)"'] \\
b \arrow[r, "g"'] & c \arrow[r, "h"'] & T(a) \arrow[r, "-T(f)"'] & T(b)
\end{tikzcd}$$

In particular, $f \circ r = k$ and thus, $\mathrm{Hom}(x, f)(r) = k$. For all $k\colon x \to b$, we have shown that if we have $\mathrm{Hom}(x, g)(k) = 0$, then $\mathrm{Hom}(x, f)(r) = k$ for some $r\colon x \to a$. For the converse, let $k\colon x \to b$ and suppose that $\mathrm{Hom}(x, f)(r) = k$ for some $r\colon x \to a$. Then, $\mathrm{Hom}(x, g)(k) = (\mathrm{Hom}(x, g) \circ \mathrm{Hom}(x, f))(r) = \mathrm{Hom}(x, g \circ f)(r)$ and by Lemma 2.6.3, we have $\mathrm{Hom}(x, g \circ f) = \mathrm{Hom}(x, 0_{a,c})(r) = 0_{x,c}$. Thus,

$$\begin{aligned}
k \in \ker(\mathrm{Hom}(x, g)) \iff \mathrm{Hom}(x, g)(k) = 0 &\iff \mathrm{Hom}(x, f)(r) = k \text{ for some } r \\
&\iff k \in \mathrm{im}(\mathrm{Hom}(x, f))
\end{aligned}$$

We recall Remark 2.4.5 to conclude that $\ker(\mathrm{Hom}(x, g)) = \mathrm{Hom}(x, f)$. $\square$

The binary biproduct of two triangles $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ and $a' \xrightarrow{f'} b' \xrightarrow{g'} c' \xrightarrow{h'} T(a')$ is defined to be the triangle $a \oplus b \xrightarrow{f \oplus f'} b \oplus b' \xrightarrow{g \oplus g'} c \oplus c' \xrightarrow{h \oplus h'} T(a \oplus a')$.

Proposition 2.6.5. *Two triangles are distinguished triangles if and only if their biproduct is a distinguished triangle.*

Proof. Let $\mathbf{A}$ be a triangulated category. Suppose $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} T(a)$ and $a' \xrightarrow{f'} b' \xrightarrow{g'} c' \xrightarrow{h'} T(a')$ are distinguished triangles. By (TC2), there is a distinguished triangle $a \oplus a' \xrightarrow{f \oplus f'} b \oplus b' \xrightarrow{k} d \xrightarrow{r} T(a \oplus a')$. Since T is additive, $T(a \oplus a') \cong T(a) \oplus T(a')$ and, hence, we have a biproduct $(T(a \oplus a'), p_{T(a)}, p_{T(a')}, i_{T(a)}, i_{T(a')})$ of $T(a)$ and $T(a')$. By (TC5), there is a morphism u_c and a morphism $u_{c'}$ making the following diagram commute:

$$\begin{tikzcd}
a' \arrow[r, "f'"'] & b' \arrow[r, "g'"'] & c' \arrow[r, "h'"'] & T(a') \\
a \oplus a' \arrow[u, "p_{a'}"'] \arrow[r, "f \oplus f'"'] \arrow[d, "p_a"'] & b \oplus b' \arrow[u, "p_{b'}"'] \arrow[r, "k"'] \arrow[d, "p_b"'] & d \arrow[u, dotted, "u_{c'}"'] \arrow[r, "r"'] \arrow[d, dotted, "u_c"'] & T(a \oplus a') \arrow[u, "p_{T(a')}"'] \arrow[d, "p_{T(a)}"'] \\
a \arrow[r, "f"'] & b \arrow[r, "g"'] & c \arrow[r, "h"'] & T(a)
\end{tikzcd}$$

The following diagram also commutes:

$$\begin{tikzcd}
a \oplus a' \arrow[r, "f\oplus f'"'] \arrow[d, "\mathrm{id}_{a\oplus a'}"'] & b \oplus b' \arrow[r, "k"'] \arrow[d, "\mathrm{id}_{b\oplus b'}"'] & d \arrow[r, "r"'] \arrow[d, dotted, "i_c\circ u_c + i_{c'}\circ u_{c'}"'] & T(a \oplus a') \arrow[r, "T(f\oplus f')"'] \arrow[d, "\mathrm{id}_{T(a\oplus a')}"'] & T(b \oplus b') \arrow[d, "\mathrm{id}_{T(b\oplus b')}"'] \\
a \oplus a' \arrow[r, "f\oplus f'"'] & b \oplus b' \arrow[r, "g\oplus g'"'] & c \oplus c' \arrow[r, "h\oplus h'"'] & T(a \oplus a') \arrow[r, "T(f\oplus f')"'] & T(b \oplus b')
\end{tikzcd} \tag{2.6.1}$$

The image of the first row of (2.6.1) under $\mathrm{Hom}(c \oplus c', -)$ is an exact sequence by Lemma 2.6.4. Since the functor $\mathrm{Hom}(c \oplus c', -)$ is additive, the image of the second row under it consists of two biproduct summands that are exact by Lemma 2.6.4. Hence, the row is an exact sequence since the biproduct of two exact sequences is an exact sequence. Let $u = i_c \circ u_c + i_{c'} \circ u_{c'}$. By the Five Lemma [15, p. 205], $\mathrm{Hom}(c \oplus c', u)$, is an isomorphism. Therefore,

$$\begin{aligned}
u \circ \mathrm{Hom}(c \oplus c', u)^{-1}(\mathrm{id}_{c\oplus c'}) &= (\mathrm{Hom}(c \oplus c', u) \circ \mathrm{Hom}(c \oplus c', u)^{-1})(\mathrm{id}_{c\oplus c'}) \\
&= \mathrm{id}_{\mathrm{Hom}(c\oplus c', c\oplus c')}(\mathrm{id}_{c\oplus c'}) = \mathrm{id}_{c\oplus c'}\,.
\end{aligned}$$

Hence, u is a split epimorphism. A similar argument with the functor $\mathrm{Hom}(-, d)$ shows that u is a split monomorphism. Therefore, u is an isomorphism by Lemma 1.2.23. So by (TC3), the forward implication holds. Details of the reverse implication, which are similar, can be found in [18, p. 33–42]. □

Definition 2.6.6 (Category of chain complexes)**.** Let $\mathbf{A}$ be an additive category. We define the *category of chain complexes* of $\mathbf{A}$ to be the category where:

(a) Objects, which are called *chain complexes*, are sequences of morphisms $a_\bullet = (d_i^a)_{i\in\mathbb{Z}}$ of the form
$$\cdots \xrightarrow{d_{-2}^a} a_{-1} \xrightarrow{d_{-1}^a} a_0 \xrightarrow{d_0^a} a_1 \xrightarrow{d_1^a} \cdots$$
such that $d_{i+1}^a \circ d_i^a = 0$ for all $i \in \mathbb{Z}$.

(b) Morphisms $f_\bullet \colon a_\bullet \to b_\bullet$ in the category of chain complexes are sequences of morphisms $(f_i)_{i\in\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{-2}^a"'] & a_{-1} \arrow[r, "d_{-1}^a"'] \arrow[d, "f_{-1}"'] & a_0 \arrow[r, "d_0^a"'] \arrow[d, "f_0"'] & a_1 \arrow[r, "d_1^a"'] \arrow[d, "f_1"'] & \cdots \\
\cdots \arrow[r, "d_{-2}^b"'] & b_{-1} \arrow[r, "d_{-1}^b"'] & b_0 \arrow[r, "d_0^b"'] & b_1 \arrow[r, "d_1^b"'] & \cdots
\end{tikzcd}$$

Definition 2.6.7 (Homotopic)**.** Two chain complex morphisms $f_\bullet \colon a_\bullet \to b_\bullet$ and $g_\bullet \colon a_\bullet \to b_\bullet$ are *homotopic* if there is a sequence of morphisms $(h_i)_{i\in\mathbb{Z}}$ such that $\mathrm{dom}(h_i) = a_i$, $\mathrm{cod}(h_i) = b_{i-1}$, and $f_i - g_i = d^b_{i-1} \circ h_i + h_{i+1} \circ d^a_i$ for all $i \in \mathbb{Z}$. The morphisms are summarized in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d^a_{-2}"'] & a_{-1} \arrow[r, "d^a_{-1}"'] \arrow[d, "f_{-1}-g_{-1}"'] & a_0 \arrow[r, "d^A_0"'] \arrow[dl, "h_0"'] \arrow[d, "f_0-g_0"'] & a_1 \arrow[r, "d^a_1"'] \arrow[dl, "h_1"'] \arrow[d, "f_1-g_1"'] & \cdots \\
\cdots \arrow[r, "d^b_{-2}"'] & b_{-1} \arrow[r, "d^b_{-1}"'] & b_0 \arrow[r, "d^b_0"'] & b_1 \arrow[r, "d^b_1"'] & \cdots
\end{tikzcd}$$

This diagram is not necessarily commutative. This relation on the morphisms forms an equivalence relation.

Definition 2.6.8 (Homotopy category)**.** Let $\mathbf{A}$ be an additive category. We define the *homotopy category* $K(\mathbf{A})$ to be the category where:

(a) The objects are precisely chain complexes.

(b) The morphisms are the homotopy equivalence classes of the category of chain complexes.

This category is additive. When working in the homotopy category, we typically omit the equivalence class notation.

Example 2.6.9. The category $K(\mathbf{A})$ is a triangulated category with T being the functor $(d^a_i)_{i\in\mathbb{Z}} \mapsto (d^a_{i+1})_{i\in Z}$ and $(f_i)_{i\in\mathbb{Z}} \mapsto (-f_{i+1})_{i\in\mathbb{Z}}$, whose distinguished triangles are sequences isomorphic to sequences of the form:

$$a_\bullet \to b_\bullet \xrightarrow{i_{b_\bullet}} T(a_\bullet) \oplus b_\bullet \xrightarrow{p_{T(a_\bullet)}} T(a_\bullet)$$

More details can be found in [6, Definition 3.4] for the interested reader.

When the $\mathbf{A}$ is an abelian category, we can pass from $K(\mathbf{A})$ to the *derived category* by "localizing quasi-isomorphisms". The resulting category is also triangulated. If the reader possesses further interest in triangulated categories, we recommend that he or she takes a look into the derived category of an abelian category.

Definition 2.6.10 (Grothendieck group of a triangulated category)**.** Let $\mathbf{A}$ be an essentially small triangulated category. Define $N_0(\mathbf{A}) \trianglelefteq G^{\mathrm{iso}}(\mathbf{A})$ to be the (normal) subgroup generated by $\{[b] - [a] - [c] \mid a \to b \to c \to T(a)$ is a distinguished triangle$\}$. The *Grothendieck group* $G_0(\mathbf{A})$ of $\mathbf{A}$ is the quotient group $G^{\mathrm{iso}}(\mathbf{A})/N_0(\mathbf{A})$. By a common abuse of terminology, we will write $[a]$ to mean the image of the isomorphism class of $[a]$ in $G_0(\mathbf{A})$ when this will not cause confusion. It follows that for every distinguished triangle $a \to b \to c \to T(a)$, we have the relation $[b] - [a] - [c] = 0$ (or equivalently, $[b] = [a] + [c]$) in $G_0(\mathbf{A})$.

Lemma 2.6.11. *Let* $\mathbf{A}$ *be a triangulated category. For all objects* $a, b \in \mathbf{A}$*, there is a distinguished triangle* $a \to a \oplus b \to b \to T(a)$*.*

Proof. Since $a \xrightarrow{\mathrm{id}_a} a \to 0 \to T(a)$ and $0 \to b \xrightarrow{\mathrm{id}_b} b \to T(0)$ are distinguished triangles, we have a distinguished triangle $a \oplus 0 \to a \oplus b \to 0 \oplus b \to T(a) \oplus T(0)$ by Proposition 2.6.5. By (TC3), the statement holds. $\square$

Corollary 2.6.12. *The Grothendieck group of a triangulated category is a quotient of the split Grothendieck group of the underlying additive category.*

Lemma 2.6.13. *Let* $\mathbf{A}$ *be a triangulated category. Then, the following hold:*

(a) The class of the zero object, $[0]$*, is the zero element of* $G_0(\mathbf{A})$*.*

(b) For all $A, B \in \mathbf{A}$*,* $[A \oplus B] = [A] + [B]$*.*

(c) For all $A \in \mathbf{A}$*,* $[T(A)] = -[A]$ *in* $G_0(\mathbf{A})$*.*

(d) Every element of $G_0(\mathbf{A})$ *is of the form* $[A]$ *for some* $A \in \mathbf{A}$*.*

Proof. Parts (a) and (b) are immediate consequences of Corollary 2.6.12. Part (c) follows directly from (TC1) and (TC4) – that is, for every object A, we have that $A \to 0 \to T(A) \to T(A)$ is a distinguished triangle.

For (d), let x be an arbitrary element of $G_0(\mathbf{A})$. Then, without loss of generality, x is of the form

$$a_1[A_1] + \cdots + a_n[A_n] - b_1[B_1] - \cdots - b_m[B_m],$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are nonnegative integers and $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathbf{A}$. Denoting $\underbrace{A \oplus \cdots \oplus A}_{k \text{ times}}$ by A^k, we have by (b) that

$$[x] = [{A_1}^{a_1} \oplus \cdots \oplus {A_n}^{a_n} \oplus T(B_1)^{b_1} \oplus \cdots \oplus T(B_m)^{b_m}]. \qquad \square$$

Proposition 2.6.14. *Let* $\mathbf{A}$ *be a triangulated category and let* $\varphi\colon G^{\mathrm{iso}}(\mathbf{A}) \to G_0(\mathbf{A})$ *be the natural quotient group homomorphism. The Grothendieck group of* $\mathbf{A}$ *satisfies the universal property that for any abelian group* G *and any group homomorphism* $\psi\colon G^{\mathrm{iso}}(\mathbf{A}) \to G$ *satisfying the property that for every distinguished triangle* $a \to b \to c \to T(a)$*,* $\psi([b]-[a]-[c]) = 0$*, there exists a unique homomorphism* $\theta\colon G_0(\mathbf{A}) \to G$ *such that* $\theta \circ \varphi = \psi$*. This statement can be summarized by the following commutative diagram:*

$$\begin{tikzcd} G^{\mathrm{iso}}(\mathbf{A}) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & G \\ G_0(\mathbf{A}) \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

Proof. The proof, which follows immediately from the definitions and is similar to that of the universal property of the Grothendieck group of abelian categories (Proposition 2.4.14), is left to the reader. $\square$

Proposition 2.6.15. *Let* $\mathbf{A}$ *be a triangulated category. We define a relation on the objects of* $\mathbf{A}$ *as follows:* $A \sim B$ *if and only if there exist objects* C_1, C_2, C_3 *such that*

$$A \oplus C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} T(A \oplus C_1)$$

and

$$B \oplus C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} C_3 \xrightarrow{g_3} T(B \oplus C_1)$$

are distinguished triangles. Then, the following hold:

(a) The relation $\sim$ *is an equivalence relation.*

(b) The set E *of equivalence classes* $\langle A \rangle$ *of objects in* $\mathbf{A}$ *is an abelian group with* $\langle A \rangle + \langle B \rangle = \langle A \oplus B \rangle$*.*

(c) The groups E *and* $G_0(\mathbf{A})$ *are isomorphic.*

Proof. (a) Reflexivity and symmetry are immediately obvious. For transitivity, suppose $X, Y, Z \in \mathbf{A}$, $X \sim Y$ and $Y \sim Z$. Then, there exist objects $C_1, C_2, C_3, D_1, D_2, D_3$ giving us the following distinguished triangles:

$$X \oplus C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} T(X \oplus C_1), \tag{2.6.2}$$
$$Y \oplus C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} C_3 \xrightarrow{g_3} T(Y \oplus C_1), \tag{2.6.3}$$
$$Y \oplus D_1 \xrightarrow{h_1} D_2 \xrightarrow{h_2} D_3 \xrightarrow{h_3} T(Y \oplus C_1), \tag{2.6.4}$$
$$Z \oplus D_1 \xrightarrow{j_1} D_2 \xrightarrow{j_2} D_3 \xrightarrow{j_3} T(Z \oplus C_1). \tag{2.6.5}$$

By Lemma 2.6.5, we take the direct sum of the first and the third triangle above, and that of the second and the fourth triangle above, giving us the distinguished triangles:

$$(X \oplus C_1) \oplus (Y \oplus D_1) \to C_2 \oplus D_2 \to C_3 \oplus D_3 \to T((X \oplus C_1) \oplus (Y \oplus D_1)), \tag{2.6.6}$$
$$(Y \oplus C_1) \oplus (Z \oplus D_1) \to C_2 \oplus D_2 \to C_3 \oplus D_3 \to T((Y \oplus C_1) \oplus (Z \oplus D_1)). \tag{2.6.7}$$

Then, since distinguished triangles are closed under isomorphism, we have that

$$X \oplus (Y \oplus C_1 \oplus D_1) \to C_2 \oplus D_2 \to C_3 \oplus D_3 \to T(X \oplus (Y \oplus C_1 \oplus D_1)), \tag{2.6.8}$$
$$Z \oplus (Y \oplus C_1 \oplus D_1) \to C_2 \oplus D_2 \to C_3 \oplus D_3 \to T(Z \oplus (Y \oplus C_1 \oplus D_1)). \tag{2.6.9}$$

are also distinguished triangles and thus, $X \sim Z$.

(b) It is easily verified that the operation $\langle A \rangle + \langle B \rangle = \langle A \oplus B \rangle$ is associative, commutative, well-defined, and has $\langle 0 \rangle$ as identity. It remains to show that all elements are invertible. For any element $A \in \mathbf{A}$, we have the trivial distinguished triangle of T(A),

$$T(A) \to T(A) \to 0 \to T^2(A). \tag{2.6.10}$$

Applying (TC4) and shifting to the left gives us a distinguished triangle

$$0 \to T(A) \to T(A) \to 0. \tag{2.6.11}$$

Shifting to the left once more yields

$$A \to 0 \to T(A) \to T(A). \tag{2.6.12}$$

Taking the direct sum of (2.6.10) and (2.6.12) gives us another distinguished triangle

$$A \oplus T(A) \to T(A) \to T(A) \to T(A) \oplus T^2(A). \tag{2.6.13}$$

From (2.6.11) and (2.6.13) we see that $0 \sim A \oplus T(A)$, hence $\langle 0 \rangle = \langle A \oplus T(A) \rangle = \langle A \rangle + \langle T(A) \rangle$ and so $\langle A \rangle = -\langle T(A) \rangle$ for all $A \in \mathbf{A}$.

(c) Define a map

$$\begin{aligned} \delta \colon G_0(\mathbf{A}) &\to E, \\ [A] &\mapsto \langle A \rangle. \end{aligned}$$

Suppose that $A \to B \to C \to T(A)$ is a distinguished triangle. Then, taking the direct sum of this triangle with the trivial distinguished triangle for C yields

$$A \oplus C \to B \oplus C \to C \to T(A \oplus C),$$

and taking the direct sum of the trivial distinguished triangle for B with the left-shifted trivial distinguished triangle for C yields

$$B \to B \oplus C \to C \to T(B),$$

from which we see that $A \oplus C \sim B$ and hence $\langle A \rangle + \langle C \rangle = \langle B \rangle$. Therefore, the relations in $G_0(\mathbf{A})$ hold in E and so δ is well-defined.

We leave the straightforward verification that δ is a surjective group homomorphism to the reader. Now suppose that $\delta([A]) = 0$. Then, $A \sim 0$ so we have objects C_1, C_2, C_3 giving us the distinguished triangles

$$\begin{gathered} A \oplus C_1 \to C_2 \to C_3 \to T(A \oplus C_1), \\ C_1 \to C_2 \to C_3 \to T(C_1). \end{gathered}$$

This in turn gives us the relations

$$\begin{gathered} [C_2] - [C_1] - [C_3] = 0, \\ [C_2] - ([A] + [C_1]) - [C_3] = 0. \end{gathered}$$

Subtracting the second equation from the first yields $[A] = 0$. Therefore, δ is injective and hence an isomorphism. $\square$

Corollary 2.6.16. *Let $\mathbf{A}$ be a triangulated category. Then, for any objects $A, B \in \mathbf{A}$, the following are equivalent:*

(a) $[A] = [B]$ *in* $G_0(\mathbf{A})$.

(b) There exist objects C_1, C_2, C_3 such that

$$A \oplus C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} T(A \oplus C_1)$$

and

$$B \oplus C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} C_3 \xrightarrow{g_3} T(B \oplus C_1)$$

are distinguished triangles.

Remark 2.6.17. The notion of a triangulated category can be generalized to that of an n-angulated category for any $n \geq 3$ (the case $n = 3$ corresponds to triangulated categories), where a distinguished n-angle is defined a natural way. The Grothendieck group of an n-angulated category for any odd n, as well as all results in this section, can also be generalized in a natural way. However, for even n, some of the definitions and results differ. The interested reader is encouraged to consult [4] for details.

Chapter 3

The Grothendieck Ring

In this chapter, we outline certain categories which are equipped with bifunctors and natural isomorphisms that give ring structures to the group constructions found in Chapter 2. Then, we generalize some familiar notions from module theory to concrete categories and apply them to monoidal categories. In the final section, we will also show that abelian categories are, in fact, equivalent to a full subcategory of modules over some (not necessarily commutative) ring.

3.1 Monoidal Categories

Monoidal categories generalize the familiar concept of tensor products of modules over a commutative ring. In this section, we show that the data of certain monoidal categories naturally give rise to a ring structure on the various Grothendieck groups found in Chapter 2.

Definition 3.1.1 (Monoidal category)**.** A *monoidal category* is a hextuple $(\mathbf{C}, \otimes, e, \alpha, \lambda, \varrho)$ consisting of a category $\mathbf{C}$ equipped with a bifunctor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ (called the *tensor product*), a distinguished object e, and three natural isomorphisms α, λ, and ϱ. For any objects A, B, C, we have $\alpha_{A,B,C}\colon A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$. For any object A, we have $\lambda_A\colon e \otimes A \xrightarrow{\cong} A$ and $\varrho_A\colon A \otimes e \xrightarrow{\cong} A$. These natural isomorphisms also satisfy the *pentagon axiom* and the *triangle axiom*:

(a) For any objects A, B, C, D, the following pentagon commutes:

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B, C, D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A, B \otimes C, D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes \mathrm{id}_D"']
\end{tikzcd}$$

(b) For any objects A, B, the following triangle commutes:

$$\begin{tikzcd}
A \otimes (e \otimes B) \arrow[rr, "\alpha_{A,e,B}"] \arrow[dr, "\mathrm{id}_A \otimes \lambda_B"'] & & (A \otimes e) \otimes B \arrow[dl, "\varrho_A \otimes \mathrm{id}_B"] \\
& A \otimes B &
\end{tikzcd}$$

The *coherence diagrams* above, as stated in [15, p. 165], allow us to unambiguously write expressions of the form "$A_1 \otimes A_2 \otimes \ldots \otimes A_n$" without parentheses, asserting that any two expressions obtained from inserting parentheses in any order and inserting instances of "$\ldots \otimes e \ldots$" or "$\ldots e \otimes \ldots$" are equivalent up to natural isomorphism. A monoidal category, thus, has the structure of a monoid where equality is replaced by natural isomorphism, $\otimes$ is the binary operation, e is the identity element (given by λ and ϱ), and associativity is given by α.

Example 3.1.2. The category of sets with $(\mathbf{Set}, \times, \{*\}, \alpha, \lambda, \varrho)$ where $\times$ is the usual cartesian product and $\{*\}$ is a singleton is a monoidal category. For any sets A, B, C, α is the obvious isomorphism $A \times (B \times C) \cong (A \times B) \times C$ whilst for any set A, λ and ϱ are the obvious isomorphisms $\{*\} \times A \cong A$ and $A \times \{*\} \cong A$ respectively.

Remark 3.1.3. When e, α, λ and/or ϱ are understood from the context, we will often omit them to simplify our notation. Hence, we will often refer to a monoidal category as $(\mathbf{C}, \otimes, e)$ or $(\mathbf{C}, \otimes)$.

Examples 3.1.4. Let R be a commutative ring. Then, $(R\text{-}\mathbf{Mod}, \otimes_R, R)$ is a monoidal category. In particular:

(a) When $R = \mathbb{Z}$, we have the monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

(b) When R is a field $\mathbb{K}$, we have the monoidal category $(\mathbf{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$.

Definition 3.1.5 (Braided monoidal category)**.** A monoidal category $(\mathbf{C}, \otimes, e)$ is *braided* if it is equipped with an additional natural isomorphism γ where for any two objects A, B, we have $\gamma_{A,B} \colon A \otimes B \xrightarrow{\cong} B \otimes A$. Furthermore, γ satisfies the *hexagon axiom* – that is, for any objects A, B, C, the following hexagons commute:

(a)

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[dl, "\mathrm{id}_A \otimes \gamma_{B,C}"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
A \otimes (C \otimes B) \arrow[d, "\alpha_{A,C,B}"'] & & (A \otimes B) \otimes C \arrow[d, "\gamma_{A \otimes B, C}"] \\
(A \otimes C) \otimes B \arrow[dr, "\gamma_{A,C} \otimes \mathrm{id}_B"'] & & C \otimes (A \otimes B) \arrow[dl, "\alpha_{C,A,B}"] \\
& (C \otimes A) \otimes B &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}[column sep=small]
 & A \otimes (B \otimes C) \arrow[dl, "\mathrm{id}_A \otimes \gamma^{-1}_{C,B}"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
A \otimes (C \otimes B) \arrow[d, "\alpha_{A,C,B}"'] & & (A \otimes B) \otimes C \arrow[d, "\gamma^{-1}_{C,A\otimes B}"] \\
(A \otimes C) \otimes B \arrow[dr, "\gamma^{-1}_{C,A} \otimes \mathrm{id}_B"'] & & C \otimes (A \otimes B) \arrow[dl, "\alpha_{C,A,B}"] \\
 & (C \otimes A) \otimes B &
\end{tikzcd}$$

The collection of maps $\gamma = \{\gamma_{A,B} \mid A, B \in \mathrm{Ob}\,\mathbf{C}\}$ is called a *braiding*. It follows that the braiding is compatible with λ and ϱ. That is, the following triangles commute for any object A:

$$\begin{tikzcd}
e \otimes A \arrow[rr, "\gamma_{e,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes e \arrow[dl, "\varrho_A"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
e \otimes A \arrow[dr, "\lambda_A"'] & & A \otimes e \arrow[ll, "\gamma_{A,e}"'] \arrow[dl, "\varrho_A"] \\
 & A &
\end{tikzcd}$$

Definition 3.1.6 (Symmetric monoidal category)**.** A braided monoidal category is *symmetric* if for all $A, B \in \mathrm{Ob}\,\mathbf{C}$, we have $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A\otimes B}$.

Example 3.1.7. Any category with finite products is symmetric monoidal with $A \otimes B = A \prod B$ and any category with finite coproducts is symmetric monoidal with $A \otimes B = A \coprod B$.

Example 3.1.8. The categories discussed in Example 3.1.2 and Examples 3.1.4 can all be given the structure of a symmetric monoidal category.

Example 3.1.9. The category of representations of a quantum group with the usual tensor product has the structure of a braided monoidal category but not that of a symmetric monoidal category. A detailed discussion of this example is beyond the scope of this paper, but interested readers are encouraged to consult [12].

Definition 3.1.10 (Ring structure on $G^{\mathrm{iso}}(\mathbf{C})$)**.** Let $(\mathbf{C}, \otimes, e)$ be a monoidal category. Define a multiplication on $G^{\mathrm{iso}}(\mathbf{C})$, where $A, B \in \mathrm{Ob}\,\mathbf{C}$, as follows:

$$[A] \cdot [B] = [A \otimes B],$$

which we extend by linearity (e.g. $[A] \cdot ([B] + [C]) = [A \otimes B] + [A \otimes C]$.) This multiplication makes $(G^{\mathrm{iso}}(\mathbf{C}), \cdot)$ a monoid with identity element $[e]$ and with associativity given by α (that is, since $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, we have that $[A] \cdot ([B] \cdot [C]) = [A \otimes (B \otimes C)] = [(A \otimes B) \otimes C] = ([A] \cdot [B]) \cdot [C]$.) It is easily verified that the distributivity axioms of a ring are also satisfied. Hence, $G^{\mathrm{iso}}(\mathbf{C})$ is a ring.

Definition 3.1.11 (Biadditive bifunctor)**.** Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be preadditive categories. We say that a bifunctor $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is *biadditive* if for every $C \in \mathbf{C}$ the functor

$$\begin{aligned} C \otimes -\colon \mathbf{D} &\to \mathbf{E}, \\ D &\mapsto \otimes(C \times D), \end{aligned}$$

is additive, and for every $D \in \mathbf{D}$ the functor

$$\begin{aligned} - \otimes D\colon \mathbf{C} &\to \mathbf{E}, \\ C &\mapsto \otimes(C \times D). \end{aligned}$$

is additive.

Example 3.1.12. By Lemma 2.2.20, we have that additive functors between additive categories preserve biproducts – thus, if $(\mathbf{C}, \otimes, e)$ is a monoidal category wherein $\mathbf{C}$ is an additive category and $\otimes$ is a biadditive bifunctor, then for any objects A, B, C, we have $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ and $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$.

Proposition 3.1.13. *If* $(\mathbf{C}, \otimes, e)$ *is a monoidal category wherein* $\mathbf{C}$ *is an additive category and* $\otimes$ *is a biadditive bifunctor, then* $N_\oplus(\mathbf{C})$ *is an ideal and hence* $G_0(\mathbf{C}, \oplus)$ *inherits the structure of a ring.*

Proof. Consider $G^{\text{iso}}(\mathbf{C})$ with the ring structure defined in Definition 3.1.10. Since we already know that $N_\oplus(\mathbf{C})$ is an additive subgroup, it remains to show that $N_\oplus(\mathbf{C})$ is invariant under left and right multiplication by elements of $G^{\text{iso}}(\mathbf{C})$.

Let $r \in G^{\text{iso}}(\mathbf{C})$ and $x \in N_\oplus(\mathbf{C})$. Then, r is of the form $\sum_{i=1}^n a_i[A_i]$ and x is of the form $\sum_{j=1}^m b_j([B_j \oplus C_j] - [B_j] - [C_j])$ where the a_i, b_j are integers and the A_i, B_j, C_j are objects in $\mathbf{C}$. It follows from Example 3.1.12 that:

$$\begin{aligned} rx &= \sum_{i=1}^n a_i[A_i] \cdot \sum_{j=1}^m b_j([B_j \oplus C_j] - [B_j] - [C_j]) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j([A_i \otimes (B_j \oplus C_j)] - [A_i \otimes B_j] - [A_i \otimes C_j]) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j([(A_i \otimes B_j) \oplus (A_i \otimes C_j)] - [A_i \otimes B_j] - [A_i \otimes C_j]). \end{aligned}$$

Hence, $rx \in N_\oplus(\mathbf{C})$. Similarly, one can show that $xr \in N_\oplus(\mathbf{C})$. Thus, $N_\oplus(\mathbf{C})$ is an ideal of $G^{\text{iso}}(\mathbf{C})$, so $G_0(\mathbf{C}, \oplus)$ is a quotient ring of $G^{\text{iso}}(\mathbf{C})$. $\square$

Example 3.1.14. Consider $(\mathbf{FinVect}_\mathbb{K}, \oplus, 0)$ where $\oplus$ is the usual direct sum and 0 is the zero vector space over $\mathbb{K}$. It is clear that $\oplus$ does not preserve biproducts (it *is* the biproduct) – that is, for arbitrary $\mathbb{K}$-vector spaces V_1, V_2, V_3, $V_1 \oplus (V_2 \oplus V_3) \ncong (V_1 \oplus V_2) \oplus (V_1 \oplus V_3)$. Therefore, with $\oplus$ as the tensor product, $G_0(\mathbf{FinVect}_\mathbb{K}, \oplus)$ does not inherit a ring structure.

Definition 3.1.15 (Left/right short exact sequence)**.** Let $\mathbf{C}$ be an abelian category. Then, a *left short exact sequence* is an exact sequence

$$0 \to A \to B \to C,$$

and a *right short exact sequence* is an exact sequence

$$A \to B \to C \to 0.$$

Definition 3.1.16 (Left/right exact functor)**.** A functor F between abelian categories is *left exact* if for every left short exact sequence $0 \to A \to B \to C$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact. We define *right exact* functors analogously.

Definition 3.1.17 (Exact functor)**.** A functor F between abelian categories is *exact* if it preserves short exact sequences. That is, for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact. It is easy to see that a functor is exact if and only if it is both left exact and right exact.

Definition 3.1.18 (Biexact bifunctor)**.** Let $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ be a bifunctor. Then, $\otimes$ is *biexact* if for every $C \in \mathbf{C}$ and $D \in \mathbf{D}$ the functors $C \otimes -$ and $- \otimes D$ (as defined in Definition 3.1.11) are exact.

Example 3.1.19. If $(\mathbf{C}, \otimes)$ is a monoidal category and $\otimes$ is biexact, then for any short exact sequence $0 \to A \to B \to C \to 0$ and any $X \in \mathbf{C}$, the sequences

$$0 \to A \otimes X \to B \otimes X \to C \otimes X \to 0$$

and

$$0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$$

are exact.

Proposition 3.1.20. *If $(\mathbf{C}, \otimes, e)$ is a monoidal category wherein $\mathbf{C}$ is an abelian category and $\otimes$ is a biexact bifunctor, then $N_0(\mathbf{C})$ is an ideal and hence $G_0(\mathbf{C})$ inherits the structure of a ring.*

Proof. Consider $G^{\text{iso}}(\mathbf{C})$ with the ring structure defined in Definition 3.1.10. Since we already know that $N_0(\mathbf{C})$ is an additive subgroup, it remains to show that $N_0(\mathbf{C})$ is invariant under left and right multiplication by elements of $G^{\text{iso}}(\mathbf{C})$.

Let $r \in G^{\text{iso}}(\mathbf{C})$ and $x \in N_0(\mathbf{C})$. Then, r is of the form $\sum_{i=1}^{n} a_i[R_i]$ and x is of the form $\sum_{j=1}^{m} b_j([B_j] - [A_j] - [C_j])$ where the a_i, b_j are integers and the R_i, A_j, B_j, C_j are objects in $\mathbf{C}$ and where $0 \to A_j \to B_j \to C_j \to 0$ is a short exact sequence for all $1 \leq j \leq m$. We have that:

$$\begin{aligned} rx &= \sum_{i=1}^{n} a_i[R_i] \cdot \sum_{j=1}^{m} b_j([B_j] - [A_j] - [C_j]) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j([R_i \otimes B_j] - [R_i \otimes A_j] - [R_i \otimes C_j]) \end{aligned}$$

Hence, $rx \in N_0(\mathbf{C})$. Similarly, one can show that $xr \in N_0(\mathbf{C})$. Thus, $N_0(\mathbf{C})$ is an ideal of $G^{\text{iso}}(\mathbf{C})$, so $G_0(\mathbf{C})$ is a quotient ring of $G^{\text{iso}}(\mathbf{C})$. $\square$

Definition 3.1.21 (Grothendieck ring)**.** When the conditions of Proposition 3.1.20 are satisfied, we call the resulting ring a *Grothendieck ring*.

Example 3.1.22. A priori, for some arbitrary commutative ring R, the tensor product $\otimes_R$ is not necessarily biexact, so $G_0(R\text{-}\mathbf{Mod})$ does not necessarily inherit a ring structure in the natural way that we have just discussed. On the other hand, it is known that the tensor product of modules distributes over the direct sum (up to isomorphism), so $G_0(R\text{-}\mathbf{Mod}, \oplus)$ *does* always inherit a ring structure.

Lemma 3.1.23. *If $(\mathbf{C}, \otimes, e)$ is a braided monoidal category, then the $G^{\text{iso}}(\mathbf{C})$ is a commutative ring. Similarly, if $\mathbf{C}$ is an additive category and $\otimes$ is a biadditive bifunctor (respectively, if $\mathbf{C}$ is an abelian category and $\otimes$ is a biexact bifunctor), then $G_0(\mathbf{C}, \oplus)$ (respectively, $G_0(\mathbf{C})$) is a commutative ring.*

Proof. The proof, which follows directly from the definitions and previous propositions, is left to the reader. $\square$

Remark 3.1.24. If $(\mathbf{C}, \otimes, e)$ is a symmetric monoidal category, then the $G^{\text{iso}}(\mathbf{C})$ (respectively, $G_0(\mathbf{C}, \oplus)$ if $\mathbf{C}$ is additive and $G_0(\mathbf{C})$ if $\mathbf{C}$ is abelian) has the additional structure of a λ-*ring*, which is a commutative ring equipped with a sequence of maps that behave like the exterior products of a vector space. The precise definition of a λ-ring is quite technical and beyond the scope of this paper, but interested readers are encouraged to consult [24].

Definition 3.1.25 (Triangulated functor)**.** Let $(\mathbf{C}, T, D)$ and $(\mathbf{D}, T', D')$ be triangulated categories. A functor $F \colon \mathbf{C} \to \mathbf{D}$ is *triangulated* if for every distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ in D, $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(T(A))$ is a distinguished triangle in D'.

Definition 3.1.26 (Monoidal triangulated category)**.** Let $(\mathbf{C}, T, D)$ be a triangulated category. Then $\mathbf{C}$ is *monoidal triangulated* if $\mathbf{C}$ also admits a symmetric monoidal structure $(\mathbf{C}, \otimes, e, \alpha, \lambda, \varrho, \gamma)$ satisfying the following conditions:

(a) There exist natural isomorphisms $\ell \colon (- \otimes T(-)) \xrightarrow{\cong} T(- \otimes -)$ and $r \colon (T(-) \otimes -) \xrightarrow{\cong} T(- \otimes -)$.

(b) For every $A \in \mathbf{C}$, the functors $(A \otimes -)$ and $(- \otimes A)$ are triangulated functors, with the corresponding natural isomorphisms ℓ and r, respectively.

(c) For every $A \in \mathbf{C}$, the following triangles commute:

$$\begin{tikzcd}
e \otimes T(A) \arrow[rr, "\lambda_{T(A)}"] \arrow[dr, "\ell_{e,A}"'] & & T(A) \\
& T(e \otimes A) \arrow[ur, "T(\lambda_A)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T(A) \otimes e \arrow[rr, "\rho_{T(A)}"] \arrow[dr, "r_{A,e}"'] & & T(A) \\
& T(A \otimes e) \arrow[ur, "T(\varrho_A)"'] &
\end{tikzcd}$$

(d) For every $A, B \in \mathbf{C}$, the following square anticommutes:

$$\begin{tikzcd} T(A) \otimes T(B) \arrow[r, "r_{A,B}"] \arrow[d, "\ell_{A,B}"'] & T(A \otimes T(B)) \arrow[d, "T(\ell_{A,B})"] \\ T(T(A) \otimes B) \arrow[r, "T(r_{A,B})"'] & T^2(A \otimes B) \end{tikzcd}$$

That is, $T(\ell_{A,B}) \circ r_{A,B} = -T(r_{A,B}) \circ \ell_{A,B}$.

Proposition 3.1.27. *Let $\mathbf{C}$ be a monoidal triangulated category wherein $\otimes$ is biadditive. Then the Grothendieck group $G_0(\mathbf{C})$ (here we are considering the Grothendieck group of $(\mathbf{C}, T, D)$ as a triangulated category, not that of the category $\mathbf{C}$ which may be abelian) inherits the structure of a commutative ring.*

Proof. We define the multiplication on $G^{\text{iso}}(\mathbf{C})$ by $[A]\cdot[B] = [A \otimes B]$ as in the case of additive and abelian categories with a monoidal structure. We verify directly that the multiplication in $G_0(\mathbf{C})$ is well-defined.

Let $A, B, C \in \mathbf{C}$ and suppose that $[B] = [C]$ in $G_0(\mathbf{C})$. By Corollary 2.6.16, we have objects D_1, D_2, D_3 giving us the following distinguished triangles:

$$B \oplus D_1 \to D_2 \to D_3 \to T(B \oplus D_1),$$
$$C \oplus D_1 \to D_2 \to D_3 \to T(C \oplus D_1).$$

Hence, tensoring these sequences with A on the left yields two new distinguished triangles

$$A \otimes (B \oplus D_1) \to A \otimes D_2 \to A \otimes D_3 \to T(A \otimes (B \oplus D_1)),$$
$$A \otimes (C \oplus D_1) \to A \otimes D_2 \to A \otimes D_3 \to T(A \otimes (C \oplus D_1)).$$

Since $\otimes$ is biadditive and the class of distinguished triangles is closed under isomorphism, we have that

$$(A \otimes B) \oplus (A \otimes D_1) \to A \otimes D_2 \to A \otimes D_3 \to T((A \otimes B) \oplus (A \otimes D_1)),$$
$$(A \otimes C) \oplus (A \otimes D_1) \to A \otimes D_2 \to A \otimes D_3 \to T((A \otimes C) \oplus (A \otimes D_1)),$$

are distinguished triangles, from which Corollary 2.6.16 gives us that $[A \otimes B] = [A \otimes C]$. Thus, we have $[A] \cdot [B] = [A] \cdot [C]$. The proof that left-multiplication is well-defined is analogous. $\square$

Remark 3.1.28. Naturally, if we drop γ and do not require the monoidal structure on $\mathbf{C}$ to be symmetric/braided, $G_0(\mathbf{C})$ will be a (not necessarily commutative) ring. However, we have chosen to be consistent with how "monoidal triangulated category" is defined in the literature.

3.2 Concrete Categories

In this section, we discuss common properties and objects between some *concrete categories* with which we are already familiar, concluding with a result about the Grothendieck ring of the category of vector spaces.

Definition 3.2.1 (Concrete category)**.** A *concrete category* is a pair $(\mathbf{C}, U)$ where $\mathbf{C}$ is a category and $U\colon \mathbf{C} \to \mathbf{Set}$ is a faithful functor. We call U the *forgetful* (or *underlying*) functor. When U is clear from context, we may simply write $\mathbf{C}$ for a concrete category $(\mathbf{C}, U)$. We also write $|C|$ for the underlying object $U(C)$ of an object $C \in \mathbf{C}$.

Example 3.2.2. The categories in Examples 1.1.10 are concrete categories.

Remark 3.2.3. It is possible to consider concrete categories over some arbitrary category $\mathbf{X}$ which is not necessarily $\mathbf{Set}$. For instance, we can consider the category of topological vector spaces as a concrete category over the category of vector spaces *or* the category of topological spaces. In this situation, concrete categories over $\mathbf{Set}$ are called *constructs* to distinguish them from arbitrary concrete categories, but we will continue to simply say "concrete category" to mean concrete categories over $\mathbf{Set}$. We refer the interested reader to [1, Section 5] for more details.

Definition 3.2.4 (Structural/universal arrow, free object)**.** Let $(\mathbf{C}, U)$ be a concrete category and let $X \in \mathbf{Set}$. A *structural arrow* with domain X is a pair (f, C) consisting of an object $C \in \mathbf{C}$ and a morphism $f\colon X \to |C| \in \operatorname{Mor}\mathbf{Set}$.

A *universal arrow* over X is a structural arrow (u, A), $u\colon X \to |A|$ for some $A \in \mathbf{C}$ which satisfies the universal property that for each $B \in \mathbf{C}$ and every structural arrow $f\colon X \to |B|$, there exists a unique $\mathbf{C}$-morphism $\theta\colon A \to B$ such that $f = u \circ U(\theta)$. This statement is summarized by the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & {|B|} \\ {|A|} \arrow[ur, dotted, "U(\theta)"'] & \end{tikzcd}$$

An object $C \in \mathbf{C}$ is *free* over X if there exists a universal arrow (u, C), $u\colon X \to |C|$.

Examples 3.2.5. (a) In any concrete category, an object A is free over $\varnothing$ if and only if A is an initial object.

(b) In R-**Mod**, free objects correspond to the usual notion of free modules; that is, an object A is free over any set which is a basis for A.

(c) In **Mon**, an object X^* is free over a set X if and only if X^* is the free monoid over X (i.e. the elements of X^* are finite sequences of members of X, including the empty sequence, and the monoid operation is concatenation).

(d) In **CRing**, the polynomial ring $\mathbb{Z}[M]$ over a set M of indeterminates is a free object over M.

Lemma 3.2.6. *Let $(\mathbf{C}, U)$ be a concrete category and let $X \in \mathbf{Set}$, $A \in \mathbf{C}$. Suppose that (u, A), $u\colon X \to |A|$ is a universal arrow. Then, for any $B \in \operatorname{Ob}\mathbf{C}$, if $f\colon A \to B$ and $g\colon A \to B$ satisfy that $U(f) \circ u = U(g) \circ u$, then $U(f) = U(g)$.*

Proof. Consider the commutative diagrams:

$$
\begin{tikzcd}
X \arrow[r, "U(f)\circ u"] \arrow[d, "u"'] & {|B|} \\
{|A|} \arrow[ur, dotted, "U(f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "U(g)\circ u"] \arrow[d, "u"'] & {|B|} \\
{|A|} \arrow[ur, dotted, "U(g)"']
\end{tikzcd}
$$

Suppose that $U(f) \neq U(g)$. Then, $f \neq g$. Since the morphisms f, g which make these diagrams commute are unique and depend on the chosen structural arrow $X \to |B|$, this means that $n \neq m$ and so $U(f) \circ u \neq U(g) \circ u$. Thus, by contrapositive, $U(f) \circ u = U(g) \circ u \implies U(f) = U(g)$. ☐

Remark 3.2.7. The preceding lemma does *not* assert that u is an epimorphism in **Set**. The cancellative property applies only to morphisms and objects in **Set** that are the image of some morphisms and objects in **C** under U!

Definition 3.2.8 (Projective object)**.** Let **C** be a category. An object $P \in \mathbf{C}$ is *projective* if for all objects $A, B \in \mathbf{C}$, all epimorphisms $e\colon A \to B$ and all morphisms $f\colon P \to B$, there exists a morphism $\theta\colon P \to A$ such that $f = e \circ \theta$. Note that we do not require θ to be unique. This statement is summarized by the following commutative diagram:

$$
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, dotted, "\theta"'] & B \\
A \arrow[ur, two heads, "e"']
\end{tikzcd}
$$

We say that f *lifts across* e.

Lemma 3.2.9. *In* **Set***, a morphism is epic if and only if it is a surjective function.*

Proof. We have the forward implication by Remark 1.2.17 and now prove the reverse implication. Let $A, B \in \mathbf{Set}$ and let $e\colon A \to B \in \operatorname{Mor}\mathbf{Set}$ be an epimorphism. Define a map:

$$
\begin{aligned}
g\colon B &\to \{0, 1\}, \\
b &\mapsto 1,
\end{aligned}
$$

and a map $h\colon B \to \{0, 1\}$ by:

$$
h(b) = \begin{cases} 0 & \text{if } b \notin \operatorname{im}(e), \\ 1 & \text{if } b \in \operatorname{im}(e). \end{cases}
$$

Then, we have $g \circ e = h \circ e$ and since e is epic, $g = h$ whence all elements of B are in $\operatorname{im}(e)$. ☐

Proposition 3.2.10. *Let $(\mathbf{C}, U)$ be a concrete category wherein U preserves epimorphisms, $X \in \mathbf{Set}$, and let $F_X \in \mathbf{C}$ be a free object over X. Then, F_X is projective.*

Proof. Let $A, B \in \mathbf{C}$, $f\colon F_X \to B$ and $e\colon A \to B$ be morphisms wherein e is epic. Let $u\colon X \to |F_X|$ be a universal arrow. Since U preserves epimorphisms, $U(e)$ is a surjective function by Lemma 3.2.9. For every $b \in (U(f) \circ u)(X) \subseteq |B|$, we choose some $a \in |A|$ such that $U(e)(a) = b$. Define a map:

$$\begin{aligned} g\colon (U(f) \circ u)(X) &\to |A|, \\ b &\mapsto a. \end{aligned}$$

This gives us a map $g \circ U(f) \circ u\colon X \to |A|$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "g \circ U(f) \circ u"'] & {|F_X|} \arrow[r, "U(f)"] & {|B|} \\ & {|A|} \arrow[ur, "U(e)"'] & \end{tikzcd}$$

By the universal property of the universal arrow, there is a morphism $\ell\colon F_X \to A$ such that $U(\ell) \circ u = g \circ U(f) \circ u$. Thus, $U(e) \circ U(\ell) \circ u = U(e \circ \ell) \circ u = U(f) \circ u$ and since u can be cancelled by Lemma 3.2.6, we have $U(f) = U(e \circ \ell)$. Since U is faithful, it is injective on $\mathrm{Hom}_{\mathbf{C}}(F_X, B)$ and therefore we have $f = e \circ \ell$, as desired. $\square$

Example 3.2.11. With U as the usual forgetful functor, the condition that U preserves epimorphisms holds in many familiar concrete categories, including **Set**, **Rel**, **Grp**, **Ab**, R-**Mod**, and $\mathbf{Vect}_{\mathbb{K}}$. A concrete category where U does not preserve epimorphisms is **Mon**. The monoid homomorphism $i\colon \mathbb{N} \to \mathbb{Z}$ whose underlying function is the inclusion map is clearly not surjective, but is epic. To see this, suppose that g_1 and g_2 are monoid homomorphisms $\mathbb{Z} \to M$ for some monoid M such that $g_1 \neq g_2$. Then, for some $x \in \mathbb{Z}$, $g_1(x) \neq g_2(x)$ and so $g_1(-x) \neq g_2(-x)$. Since either x or $-x$ must be in $\mathbb{N}$, $g_1 \circ i \neq g_2 \circ i$. Therefore, by contrapositive, we have $g_1 \circ i = g_2 \circ i \implies g_1 = g_2$.

Definition 3.2.12 (Flat object)**.** Let $(\mathbf{C}, \otimes, e)$ be a monoidal category. A *flat object* of $(\mathbf{C}, \otimes, e)$ is an object $X \in \mathbf{C}$ such that $X \otimes -$ and $- \otimes X$ are exact functors.

Example 3.2.13. In R-**Mod**, projective and flat objects correspond respectively to projective and flat modules.

The following example shows that the converse of Proposition 3.2.10 is false.

Example 3.2.14. Let $R = \mathbb{Z}_6$ and consider $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ as $\mathbb{Z}_6$-modules in the natural way (i.e. consider $\mathbb{Z}_2 = \{\bar{0}, \bar{3}\}$ and $\mathbb{Z}_3 = \{\bar{0}, \bar{2}, \bar{4}\}$ as submodules of Z_6). Then, in $\mathbb{Z}_6$-**Mod**,

$\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective but not free. The map

$$\begin{aligned}
\phi\colon \mathbb{Z}_6 &\to \mathbb{Z}_2 \oplus \mathbb{Z}_3 \\
\bar{0} &\mapsto (\bar{0}, \bar{0}) \\
\bar{1} &\mapsto (\bar{3}, \bar{2}) \\
\bar{2} &\mapsto (\bar{0}, \bar{4}) \\
\bar{3} &\mapsto (\bar{3}, \bar{0}) \\
\bar{4} &\mapsto (\bar{0}, \bar{2}) \\
\bar{5} &\mapsto (\bar{3}, \bar{4})
\end{aligned}$$

is a $\mathbb{Z}_6$-module isomorphism and it is clear that $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module. The fact that $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are not free follows from the fact that any free $\mathbb{Z}_6$-module admits a basis and so must either be infinite or contain $6x$ elements for some nonnegative integer x.

Proposition 3.2.15. *In $(R\text{-}\mathbf{Mod}, \otimes_R, R)$, projective objects are flat. That is, projective modules are flat modules (and hence, free modules are flat modules).*

Proof. We refer the reader to [11, Section 5.4] for the details. □

Remark 3.2.16. In general, projective objects need not be flat. One such example is the category $\mathfrak{M}_G$ of Mackey functors for some group G. A detailed discussion is beyond the scope of this paper, but interested readers are encouraged to consult [14].

Example 3.2.17. Since for any field $\mathbb{K}$, every $\mathbb{K}$-vector space is a free $\mathbb{K}$-module, it follows that every object in $(\mathbf{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ is flat. Thus, $\otimes_{\mathbb{K}}$ is biexact and $G_0(\mathbf{Vect}_{\mathbb{K}})$ inherits a ring structure.

3.3 Modules & Algebras over a Ring

In this final section, we will discuss how modules relate to abelian categories in a general way and introduce the advanced reader to some ideas that are natural points of further study. While many results and even definitions are abridged for brevity, full details can be found by following references. We begin with a theorem that asserts that, while quite abstractly and seemingly arbitrarily defined, abelian categories are, in fact, equivalent to concrete categories of modules over some ring. The precise statement is as follows:

Theorem 3.3.1 (Freyd-Mitchell Theorem)**.** *Let $\mathbf{C}$ be an essentially small abelian category. There exists a ring R (with unity, not necessarily commutative) and a fully faithful exact functor $F\colon \mathbf{C} \to R\text{-}\mathbf{Mod}$. (Thus far in the paper, we have always assumed R to be commutative, so it is important to note here that R-$\mathbf{Mod}$ denotes the category of all* left *R-modules.)*

Proof. A full proof of this theorem can be found in [8, Section 7.3] (note that Freyd's terminology differs somewhat from the modern terminology – what he calls a "fully abelian" category is simply an abelian category with a fully faithful exact functor to the category of R-modules). We will give a sketch of the proof.

Let $\mathbf{LX}$ be the category of left exact functors from $\mathbf{C}$ to $\mathbf{Ab}$. We construct a contravariant embedding $H\colon \mathbf{C} \to \mathbf{LX}$ which sends every $C \in \mathbf{C}$ to the covariant hom-functor $\mathrm{Hom}_{\mathbf{C}}(C, -)$. By the Yoneda Lemma ([15, p. 61]), H is fully faithful and by [8, Theorem 7.33], H is exact. By [8, Theorems 7.31-7.32], $\mathbf{LX}$ is abelian and has an object

$$I = \prod_{C \in \mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(C, -)$$

called its *injective cogenerator*.

Let $R = \mathrm{Hom}_{\mathbf{LX}}(I, I)$. Now define $G\colon \mathbf{LX} \to R\text{-}\mathbf{Mod}$ to be another contravariant, fully faithful exact functor which sends each left exact functor $B\colon \mathbf{C} \to \mathbf{Ab}$ to $\mathrm{Hom}_{\mathbf{LX}}(B, I)$. The composite $G \circ H\colon \mathbf{C} \to R\text{-}\mathbf{Mod}$ is the functor F in the statement of the theorem. $\square$

Definition 3.3.2 (Split short exact sequence)**.** Let $\mathbf{C}$ be an abelian category. A short exact sequence

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$$

is said to *split* if there exists $f'\colon b \to a$ and $g'\colon c \to b$ such that $f' \circ f = \mathrm{id}_a$ and $g' \circ g = \mathrm{id}_b$.

Proposition 3.3.3. *Let $\mathbf{C}$ be an abelian category. The following are equivalent:*

(a) The exact sequence $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ is split.

(b) $b \cong a \oplus c$

Proof. By the Freyd-Mitchell Theorem, we employ the splitting lemma for modules (see [2, Theorem 3.9]). $\square$

We will henceforth only consider left modules for convenience, though all proofs can be analogously applied to right modules. For every ring R, the category $R\text{-}\mathbf{Mod}^{\mathrm{fg}}$ of finitely generated left R-modules is an abelian category. The subcategory $R\text{-}\mathbf{pMod}^{\mathrm{fg}}$ of finitely generated left projective R-modules is also an abelian category. Biproducts are simply direct sum. For each ring, there are two Grothendieck groups of particular interest to the area of K-Theory:

Definition 3.3.4. Let R be a ring. We define the following two Grothendieck groups:

$$G_0(R) := G_0(R\text{-}\mathbf{Mod}^{\mathrm{fg}}) \qquad\qquad K_0(R) := G_0(R\text{-}\mathbf{pMod}^{\mathrm{fg}})$$

Lemma 3.3.5. *Let R be a ring. The Grothendieck group of R-$\mathbf{pMod}^{\mathrm{fg}}$ is equal to the split Grothendieck group of R-$\mathbf{pMod}^{\mathrm{fg}}$. That is, we have $K_0(R) = G_0(R\text{-}\mathbf{pMod}^{\mathrm{fg}}, \oplus)$.*

Proof. Any sequence of projective modules splits (see [2, Theorem 5.1]). By Proposition 3.3.3, the equality holds. $\square$

If R is a principle ideal domain, all projective modules are free (see [2, Corollary 6.3]).

Example 3.3.6. Detailed definitions of the italicized terms in this example can be found in [17, pp. 1–7] for the reader not already familiar with them.

An *associative algebra* is a structure which is simultaneously a ring and a module over a (possibly different) ring in such a way that the ring multiplication is bilinear with respect to the module multiplication. A *coalgebra* is the categorical dual of an associative algebra, and a *bialgebra* is a structure which is simultaneously an associative algebra and a coalgebra that satisfies some additional compatibility axioms.

The category of modules over a bialgebra has the structure of a monoidal category, and a certain class of bialgebras, called *Hopf algebras*, have the structure of a rigid monoidal category (monoidal categories where all objects are *dualizable*, a generalization of the dual space of a vector space in $\mathbf{Vect}_{\mathbb{K}}$—see [3, Example 2.1.4] for details).

By [3, Proposition 2.1.8], the tensor product in a rigid monoidal category is biexact, hence the category of modules over a Hopf algebra is another example of a category with a well-defined Grothendieck ring.

For readers who would like to know more, the following lemma is Exercise 9 in [16, Section 6D].

Lemma 3.3.7. *For any ring R and nilpotent ideal I, we have $K_0(R) \cong K_0(R/I)$.*

If R is commutative, then its *nilradical* — the collection of all nilpotent elements — is a nilpotent ideal and, thus, the quotient ring is an integral domain. This is nice because one can now restrict the study of K_0 of commutative rings, to K_0 of integral domains.

Index

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