Morita Equivalence

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Preface

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Introduction

The theory of modules over rings dates back to the late 19th century, and they were used to great success in an 1882 paper by Richard Dedekind and Anton Weber, [DW82], which was concerned with algebraic geometry. Rings, however, had not undergone axiomatic development until around 40 years later, at the hands of Emmy Noether and Wolfgang Krull.

Categories, on the other hand, did not appear in the literature until 1945, when Samuel Eilenberg and Saunders Mac Lane published their paper [EM45]. While the original motivation for categories, functors, and natural transformations were certain topics of group theory and topology, category theory has become an immensely useful and prevalent tool in modern algebra, and is also studied today for its own sake.

A natural question concerning rings is: do the modules of a ring contain all the information about the structure of that ring? The answer to this question is "no." This leads to a new sense of "equivalence" of rings which is coarser than isomorphism. That is, two rings are "equivalent" when their categories of modules are equivalent (in the category-theoretical sense). Japanese mathematician Kiiti Morita is known for having stated and proved a theorem which gave an equivalent condition for this equivalence in an influential 1958 paper, [Mor58]. This theorem plays an important role in modern algebra, and in his honour, this equivalence of module categories between two rings has been dubbed "Morita equivalence."

In this paper, we begin by reviewing the material necessary to define Morita equivalence, and we examine two classical examples of Morita equivalence. We then delve into what is now called "Morita theory", which is in regards to the key theorems of Morita in [Mor58], the results leading up to them, and their immediate consequences. We continue on by introducing bicategories, with a focus on the bicategory of bimodules. We then look at ways in which this bicategorical perspective can be used in Morita theory, including the neat encapsulation such a perspective provides to the concept of Morita equivalence. Starting in the second chapter, we introduce the representation theory of groups, another old, yet current area of mathematics, and we examine how this connects with module theory. We then use this connection to identify some examples of Morita equivalence in representation theory, particularly in the case of representation-theoretic dualities of groups.

1 Morita Equivalence

1.1 Category Theory: Basic Definitions and Theorems

We begin by stating many of the basic category-theoretic notions used throughout this paper. Readers experienced in category theory can safely skip this section.

Definition 1.1.1 (Category). A *category* $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* between A and B;
- for every three objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ an associative binary operation

  $$\circ\colon\ \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called *composition*; and
- for every object $A \in \mathrm{Ob}(\mathcal{C})$, a morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ called the *identity morphism* on A such that for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ where $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have

  $$\mathrm{id}_Y \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_X = f.$$

Definition 1.1.2 (Functor). Given categories $\mathcal{C}$ and $\mathcal{C}'$, a *functor* F from $\mathcal{C}$ to $\mathcal{C}'$, denoted $F\colon \mathcal{C} \to \mathcal{C}'$, consists of

- a function $F\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$; and
- for every two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a function

  $$F\colon\ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B)),$$

which must satisfy the following axioms:

- given an object $A \in \mathcal{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$; and
- given objects $A, B, C \in \mathcal{C}$ and morphisms $f\colon A \to B$ and $g\colon B \to C$, we have $F(g \circ f) = F(g) \circ F(f)$.

Definition 1.1.3 (Natural transformation)**.** Given categories $\mathcal{C}$ and $\mathcal{C}'$, and functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C} \to \mathcal{C}'$, a *natural transformation* α from F to G, denoted $\alpha\colon F \to G$, is a family of maps

$$(\alpha_A\colon F(A) \to G(A))_{A\in\mathcal{C}}$$

such that for $A, A' \in \mathcal{C}$ and $f\colon A \to A'$, we have

$$G(f) \circ \alpha_A = \alpha_{A'} \circ F(f).$$

It is also said that α is natural in A.

If α_A is an isomorphism for every $A \in \mathcal{C}$, then α is called a *natural isomorphism*.

Definition 1.1.4 (Equivalence of categories)**.** Given categories $\mathcal{C}$ and $\mathcal{C}'$, a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called an *equivalence of categories* if there is a functor $F'\colon \mathcal{C}' \to \mathcal{C}$ and natural isomorphisms $\alpha\colon \mathrm{id}_{\mathcal{C}} \to F' \circ F$ and $\alpha'\colon \mathrm{id}_{\mathcal{C}'} \to F \circ F'$.

Definition 1.1.5 (Product category)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, the *product category* $\mathcal{C} \times \mathcal{D}$ is defined as the category with

- objects (C, D) for $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$;
- morphisms (f, g) for $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ and $g \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ for objects $C, C' \in \mathrm{Ob}(\mathcal{C})$ and $D, D' \in \mathrm{Ob}(\mathcal{D})$;
- composition defined as pointwise composition of morphisms: $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$;
- and identity objects $1_{(C,D)} = (1_C, 1_D)$ for $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$.

Definition 1.1.6 (Bifunctor)**.** A *bifunctor* $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor from the product category $\mathcal{A} \times \mathcal{B}$ to the category $\mathcal{C}$.

Definition 1.1.7 (Opposite category)**.** Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is defined as the category with

- objects C for $C \in \mathrm{Ob}(\mathcal{C})$;
- morphisms $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C', C)$;
- composition defined in reverse: $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$;
- and the same identity objects as in $\mathcal{C}$.

Definition 1.1.8 (Contravariant functor)**.** A *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. For contrast, functors which are not contravariant are sometimes called *covariant*.

Example 1.1.9. Given a category $\mathcal{C}$ and $A \in \mathrm{Ob}(\mathcal{C})$, the functor $\mathrm{Hom}_{\mathcal{C}}(A,-)\colon \mathcal{C} \to \mathsf{Set}$ is defined by

$$\begin{aligned} C &\mapsto \mathrm{Hom}_{\mathcal{C}}(A,C) \\ \mathrm{Hom}_{\mathcal{C}}(A,-)(f)\colon g &\mapsto f \circ g \end{aligned}$$

for $f\colon C \to C'$ and $g\colon A \to C$.

There is also the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,A)\colon \mathcal{C} \to \mathsf{Set}$, defined by

$$\begin{aligned} C &\mapsto \mathrm{Hom}_{\mathcal{C}}(C,A) \\ \mathrm{Hom}_{\mathcal{C}}(-,A)(f)\colon g &\mapsto g \circ f \end{aligned}$$

for $f\colon C \to C'$ and $g\colon C' \to A$.

We also have the bifunctor $\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$, defined by

$$\begin{aligned} (A,B) &\mapsto \mathrm{Hom}_{\mathcal{C}}(A,B) \\ \mathrm{Hom}_{\mathcal{C}}(-,-)(f,g)\colon h &\mapsto g \circ h \circ f \end{aligned}$$

for $f\colon A \to B$, $g\colon X \to Y$, and $h\colon B \to X$.

Definition 1.1.10 (Adjoint functors)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. If there is a natural isomorphism

$$\tau\colon\ \mathrm{Hom}_{\mathcal{D}}(F-,-) \to \mathrm{Hom}_{\mathcal{C}}(-,G-),$$

then (F,G) is called an *adjoint pair*, G is called the right adjoint of F, and F is called the left adjoint of G. Note that the condition that τ is natural means that τ is required to be natural in the elements $(A,B) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$.

Example 1.1.11. Every equivalence of categories forms an adjoint pair with its inverse. Thus, if F is an equivalence and G is its inverse, since G is also an equivalence, F is both the left and right adjoint of G, and vice versa.

Definition 1.1.12 (Subcategory)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if

- $\mathrm{Ob}(\mathcal{D})$ is a subcollection of $\mathrm{Ob}(\mathcal{C})$;
- for all $A,B \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(A,B)$ is a subcollection of $\mathrm{Hom}_{\mathcal{C}}(A,B)$;
- for all $A \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{D}}(A,A)$;
- if f,g are morphisms in $\mathcal{D}$ and the domain of g is the codomain of f, then $g \circ f$ is a morphism in $\mathcal{D}$; and
- the composition law in $\mathcal{D}$ is equal to the composition law in $\mathcal{C}$.

Definition 1.1.13 (Full subcategory)**.** Given a category $\mathcal{C}$, a subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a *full subcategory* of $\mathcal{C}$ if, for every $A,B \in \mathrm{Ob}(\mathcal{D})$,

$$\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B).$$

Definition 1.1.14 (Skeleton)**.** Given a category $\mathcal{C}$, a subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *skeleton* of $\mathcal{C}$ if each object of $\mathcal{C}$ is isomorphic to exactly one object of $\mathcal{D}$.

Proposition 1.1.15 ([ML98, p. 93])**.** *Every category $\mathcal{C}$ is equivalent to its skeletons.*

Definition 1.1.16 (Full, faithful functor)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *full* when the map of F on morphisms is surjective, and is called *faithful* if the map of F on morphisms is injective. If F is both full and faithful, then it is called *fully faithful*.

Definition 1.1.17 (Essentially surjective functor)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective* if each object $X \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to $F(A)$ for some object $A \in \mathcal{C}$.

Theorem 1.1.18 ([ML98, p. 93])**.** *Given two categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$, the following are equivalent:*

- *F is an equivalence of categories; and*
- *F is fully faithful and essentially surjective.*

1.2 Module Theory: Basic Definitions and Theorems

We recall some basic facts in module theory that are normally covered in an undergraduate course of algebra, as well as some facts that are more advanced.

Definition 1.2.1 (Ring)**.** A *ring* R consists of

- a non-empty set R;
- a binary operation "+" called addition such that $(R,+)$ is an abelian group with identity $0 \in R$; and
- a binary operation "$\cdot$" called multiplication that is associative, distributive over "+", and has an identity $1 \in R$.

Definition 1.2.2 (Left module)**.** Given a ring R, a *left R-module M* consists of

- a non-empty set M;
- a binary operation "+" such that $(M,+)$ is an abelian group with identity $0 \in M$;
- a map $R \times M \to M$ called a *left action* such that for $r,s \in R$ and $x,y \in M$, we have
  1. $r(x+y) = rx + ry$;
  2. $(r+s)x = rx + sx$;
  3. $(rs)x = r(sx)$; and
  4. $1_R x = x$.

We may write ${}_RM$ to emphasize the left R-module structure.

Definition 1.2.3 (Right module)**.** Given a ring R, a *right R-module M* consists of

- a non-empty set M;
- a binary operation "+" such that $(M,+)$ is an abelian group with identity $0 \in M$;
- a map $M \times R \to M$ called a *right action* such that for $r,s \in R$ and $x,y \in M$, we have
  1. $(x+y)r = xr + yr$;
  2. $x(r+s) = xr + xs$;
  3. $x(sr) = (xs)r$; and
  4. $x1_R = x$.

We may write M_R to emphasize the right R-module structure.

Example 1.2.4. A ring R is a left R-module and a right R-module, with left multiplication and right multiplication actions respectively. These modules are called the left and right regular modules, respectively.

Example 1.2.5. Given a ring R, the n-fold direct product R^n is a left R-module and a right R-module with respect to left and right pointwise multiplication actions respectively.

Definition 1.2.6 (Module homomorphism)**.** Given a ring R and left R-modules M, M', a map $f : M \to M'$ is called a *module homomorphism* if, for $x,y \in M$ and $r \in R$,

- $f(x+y) = f(x) + f(y)$; and
- $f(rx) = rf(x)$.

Homomorphisms of right modules are defined similarly.

We denote the category of left modules over a ring R by ${}_R\mathsf{Mod}$. Its objects are modules and its morphisms are module homomorphisms.

Definition 1.2.7 (Bilinear map)**.** Given vector spaces X, Y, and Z over a field $\mathbb{K}$, a *bilinear map* is a function $B\colon X \times Y \to Z$ such that for $x \in X$ and $y \in Y$, the map

$$x \mapsto B(x,y)$$

is a linear map $X \to Z$, and the map

$$y \mapsto B(x,y)$$

is a linear map from Y to Z.

The next seven definitions are useful in understanding one of the first examples of Morita equivalence.

Definition 1.2.8 (Algebra)**.** Given a field $\mathbb{K}$, an *(associative) algebra A* over $\mathbb{K}$ is a vector space equipped with an associative, bilinear map $A \times A \to A$ called multiplication in A.

Definition 1.2.9 (Jacobson radical)**.** Given an algebra A and an A-module M, the *Jacobson radical* $\mathrm{rad}(M)$ of M is the intersection of all maximal A-submodules of M.

Definition 1.2.10 (Artinian module)**.** Given a commutative ring R, an R-algebra A, and a A-module M, we say that M is an *artinian module* if, whenever there is a sequence

$$M \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

of A-submodules of M, then there exists $n_0 \in \mathbb{N}$ such that $M_n = M_{n_0}$ for all $n \geq n_0$.

Definition 1.2.11 (Artinian algebra)**.** Given a commutative ring R and an R-algebra A, A is said to be a left (right) *artinian algebra* if the left (right) regular module is artinian.

Definition 1.2.12 (Basic algebra)**.** Given an artinian algebra A, if for every pair of simple submodules S_1 and S_2 of $A/\mathrm{rad}(A)$ one has $S_1 \cong S_2 \implies S_1 = S_2$, then A is called a *basic algebra*.

Definition 1.2.13 (Idempotent)**.** Given a ring R, an element $x \in R$ is called an *idempotent* of R if $x \cdot x = x$.

Definition 1.2.14 (Orthogonal idempotent)**.** Given a ring R, two idempotents e_1 and e_2 are called *orthogonal idempotents* if $e_1 e_2 = e_2 e_1 = 0$. An idempotent $e \neq 0$ is *primitive* if whenever $e = e_1 + e_2$ with e_1, e_2 orthogonal idempotents, then either $e_1 = 0$ or $e_2 = 0$.

Definition 1.2.15 (Balanced map)**.** Given a right R-module M, a left R-module N, and an abelian group G, a map $f : M \times N \to G$ is *R-balanced* if

- $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$;
- $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$; and
- for $r \in R$, $f(mr, n) = f(m, rn)$.

Definition 1.2.16 (Tensor product of modules)**.** Given a right R-module M, a left R-module N, an abelian group T, and an R-balanced map $\tau \colon M \times N \to T$, we call (T, τ) a *tensor product* of M and N if for every abelian group G and every R-balanced map $f : M \times N \to G$, there is a unique group homomorphism $g \colon T \to G$ such that

$$g \circ \tau = f.$$

Proposition 1.2.17. *Given a right R-module M and a left R-module N, if (T, τ) and (T', τ') are tensor products of M and N, then T and T' are isomorphic abelian groups. In other words, tensor products are unique up to isomorphism.*

Proof. Since τ' is R-balanced and T is a tensor product, there is a unique group homomorphism $f: T \to T'$ such that

$$\tau' = f \circ \tau.$$

Since τ is R-balanced and T' is a tensor product, there is a unique group homomorphism $g: T' \to T$ such that

$$\tau = g \circ \tau'.$$

We thus obtain

$$\tau = g \circ f \circ \tau \quad \text{and} \quad \tau' = f \circ g \circ \tau',$$

which shows that f is a group isomorphism. Therefore, $T \cong T'$. $\square$

Proposition 1.2.18. *A tensor product exists for any right R-module M and left R-module N.*

Proof. Denote by X the module with basis $\{e_{m,n}: m \in M, n \in N\}$. Let Y be the subgroup of X generated by the following elements:

$$\begin{aligned} &e_{m+m',n} - e_{m,n} - e_{m',n};\\ &e_{m,n+n'} - e_{m,n} - e_{m,n'};\\ &e_{mr,n} - e_{m,rn}, \end{aligned}$$

where $m, m' \in M, n, n' \in N, r \in R$. Denote by $\iota: M \times N \to X$ the canonical injection and $\pi: X \to X/Y$ the projection. Set $\alpha = \pi \circ \iota$.

We show that α is R-balanced. However, this is equivalent to checking that the images under α of the elements defining Y are 0, but this follows automatically.

Let G be an abelian group and let $f: M \times N \to G$ be an R-balanced map. Then, there is a unique group homomorphism $h: X \to G$ defined by $h(e_{m,n}) = f(m,n)$, and it is such that $h \circ \iota = f$. We show that the generators of A are in $\ker h$.

$$h(e_{m+m',n} - e_{m,n} - e_{m',n}) = h(e_{m+m',n}) - h(e_{m,n}) - h(e_{m',n}) = f(e_{m+m',n}) - f(e_{m,n}) - f(e_{m',n}) = 0$$

$$h(e_{m,n+n'} - e_{m,n} - e_{m,n'}) = h(e_{m,n+n'}) - h(e_{m,n}) - h(e_{m,n'}) = f(e_{m,n+n'}) - f(e_{m,n}) - f(e_{m,n'}) = 0$$

$$h(e_{mr,n} - e_{m,rn}) = h(e_{mr,n}) - h(e_{m,rn}) = f(e_{mr,n}) - f(e_{m,rn}) = 0$$

This shows that the map $p: X/Y \to G$ defined by $p(e_{m,n}) = f(m,n)$ is well-defined (and unique). Finally, we obtain that $p \circ \alpha$ is the unique group homomorphism satisfying $p \circ \alpha = f$. Thus, $(X/Y, p)$ is a tensor product for M and N. $\square$

Let (T, τ) be a tensor product of a right R-module M and a left R-module N. Since tensor products are unique up to isomorphism, we use the standard notation $T = M \otimes_R N$ and $\tau(m,n) = m \otimes n$. By an abuse of notation, we call $M \otimes_R N$ *the* tensor product of M and N.

Definition 1.2.19 (Free module)**.** Given a ring R, a *free left R-module* is a left R-module that is isomorphic to the direct sum of some copies (possibly an infinite number) of R.

Proposition 1.2.20. *Given a ring R, every left R-module is a quotient of a free left R-module.*

Proof. Let M be a module with generating set B. Now let F be a free left R-module with B as a basis. Then, we construct a module homomorphism $f: F \to M$ induced by $f(b) = b$ for $b \in B$. That is, define f by

$$f\left(\sum_i r_i b_i\right) = \left(\sum_i r_i b_i\right),$$

for $r_i \in R$ and $b_i \in B$. Then f is evidently a surjection, and by the first isomorphism theorem, we have

$$F/\ker f \cong M. \qquad \square$$

Definition 1.2.21 (Simple module)**.** Given a ring R, a left R-module M is called *simple* if M has no non-zero proper submodule.

Definition 1.2.22 (Semisimple module)**.** Given a ring R, a left R-module M is called *semisimple* if M is the direct sum of simple modules.

Proposition 1.2.23. *Given a ring R, every simple module of R is isomorphic to R/M for some maximal ideal M of R, and for every maximal ideal M, R/M is isomorphic to a simple module of R.*

Proof. Given a maximal ideal M, we know that R/M has no proper ideals. Furthermore, we know that every ideal of R/M corresponds to a proper submodule of R/M as an R-module. Thus R/M is a simple module.

Let N be a simple R-module. Given $0 \neq n \in N$, the map $r \mapsto rn$ is a surjection, as Rn is a non-zero submodule of N, and thus Rn contains N. Thus $R/I \cong N$ for some ideal I of R. Now suppose I is not maximal. Then there is an ideal J of R such that $I \subsetneq J \subsetneq R$. However this means that J corresponds to some nontrivial proper ideal of $R/I \cong N$, and so N has a nonzero proper submodule. This is a contradiction, so I must be maximal. $\square$

Definition 1.2.24 (Semisimple module category)**.** Given a ring R, the category of left (right) R-module ${}_R\mathsf{Mod}$ is said to be *semisimple* if every module $M \in \mathrm{Ob}({}_R\mathsf{Mod})$ is semisimple.

Lemma 1.2.25. *Let R be a ring, and let M, N_1, and N_2 be (left) R-modules. Then*

$$\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong \mathrm{Hom}_R(M, N_1) \oplus \mathrm{Hom}_R(M, N_2)$$

as abelian groups.

Proof. We first define the projection maps

$$\pi_1 : N_1 \oplus N_2 \to N_1$$

and

$$\pi_2 \colon N_1 \oplus N_2 \to N_2$$

by $\pi_1(n_1, 0) = n_1$ and $\pi_2(n_2, 0) = n_2$. Then given $f \in \mathrm{Hom}_R(M, N_1 \oplus N_2)$, the map $\rho(f) = (\pi_1 \circ f, \pi_2 \circ f)$ is a group isomorphism $\mathrm{Hom}_R(M, N_1 \oplus N_2) \to \mathrm{Hom}_R(M, N_1) \oplus \mathrm{Hom}_R(M, N_2)$. Firstly, ρ is a homomorphism because

$$\begin{aligned}
\rho(f+g) &= (\pi_1 \circ (f+g), \pi_2 \circ (f+g)) \\
&= (\pi_1 \circ f + \pi_1 \circ g, \pi_2 \circ f + \pi_2 \circ g) \\
&= (\pi_1 \circ f, \pi_2 \circ f) + (\pi_1 \circ g, \pi_2 \circ g) \\
&= \rho(f) + \rho(g).
\end{aligned}$$

It is also the case that ρ has an inverse homomorphism. Given

$$(f, g) \in \mathrm{Hom}_R(M, N_1) \oplus \mathrm{Hom}_R(M, N_2),$$

we define $\rho^{-1}(f, g) = h$ where $h(m) = (f(m), g(m))$. Then

$$\rho(\rho^{-1}(f, g)) = \rho(h) = (\pi_1 \circ h, \pi_2 \circ h) = (f, g).$$

Also, for $f \in \mathrm{Hom}_R(M, N_1 \oplus N_2)$,

$$\rho^{-1}(\rho(f)) = \rho^{-1}(\pi_1 \circ f, \pi_2 \circ f) = f.$$

Thus ρ is a group isomorphism. □

Lemma 1.2.26 ([HS97, Proposition 3.4])**.** *Let R be a ring, and let M_1, M_2, and N be (left) R-modules. Then*

$$\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$$

as abelian groups.

Proof. We first define the embedding maps

$$\iota_1 \colon M_1 \to M_1 \oplus M_2$$

and

$$\iota_2 \colon M_2 \to M_1 \oplus M_2$$

by $\iota_1(m_1) = (m_1, 0)$ and $\iota_2(m_2) = (0, m_2)$. Then, given $f \in \mathrm{Hom}_R(M_1 \oplus M_2, N)$, the map $\sigma(f) = (f \circ \iota_1, f \circ \iota_2)$ is a group isomorphism $\mathrm{Hom}_R(M_1 \oplus M_2, N) \to \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$. Firstly, σ is a homomorphism because

$$\begin{aligned}
\sigma(f+g) &= ((f+g) \circ \iota_1, (f+g) \circ \iota_2) \\
&= (f \circ \iota_1 + g \circ \iota_1, f \circ \iota_2 + g \circ \iota_2) \\
&= (f \circ \iota_1, f \circ \iota_2) + (g \circ \iota_1, g \circ \iota_2) \\
&= \sigma(f) + \sigma(g).
\end{aligned}$$

It is also the case that σ has an inverse homomorphism. Given

$$(f,g) \in \mathrm{Hom}_R(M_1,N) \oplus \mathrm{Hom}_R(M_2,N),$$

we define $\sigma^{-1}(f,g) = h$ where $h(m_1,m_2) = f(m_1) + g(m_2)$. Then

$$\sigma(\sigma^{-1}(f,g)) = \sigma(h) = (h \circ \iota_1, h \circ \iota_2) = (f,g).$$

Also, for $f \in \mathrm{Hom}_R(M_1 \oplus M_2, N)$,

$$\sigma^{-1}(\sigma(f)) = \sigma^{-1}(f \circ \iota_1, f \circ \iota_2) = f.$$

Thus σ is a group isomorphism. □

Definition 1.2.27 (Simple ring)**.** A non-zero ring is called a *simple ring* if its only ideals are 0 and the ring itself.

We see from the definition of a simple ring and Proposition 1.2.23 that simple rings each have a unique simple module up to isomorphism.

Definition 1.2.28 (Semisimple ring)**.** A ring R is said to be *left semisimple* if R is semisimple as a left R-module. Similarly, R is *right semisimple* if R is semisimple as a right R-module.

Proposition 1.2.29 ([Lam01, Corollary 3.7])**.** *A left semisimple ring is always right semisimple, and vice versa.*

Due to this proposition, we will henceforth refer to left and right semisimple rings as semisimple. It may be of interest to the reader that this proposition is in fact a corollary of a more general version of Lemma 2.1.15.

Theorem 1.2.30 (Jacobson density, [Isa09, p. 185])**.** *Let U be a simple right R-module and write $D = \mathrm{End}_R(U)$. Let γ be any D-linear operator on U and let $X \subseteq U$ be any finite D-linear independent subset. Then there exists an element $r \in R$ such that $xr = \gamma(x)$ for all $x \in X$.*

1.3 Morita Equivalence: Definition and Examples

We now present the definition of Morita equvialence and we follow up with two classical examples.

Definition 1.3.1 (Morita equivalence)**.** Two rings R and S are called *Morita equivalent* if ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ are equivalent categories.

It is evident that if two rings are isomorphic, they are Morita equivalent. However, the converse in general is not true. We will see in Example 1.3.3 two rings that are not isomorphic, yet are Morita equivalent. However, the following partial converse holds.

Theorem 1.3.2 ([Lam99, p. 494]). *If two rings are Morita equivalent, then their centers are isomorphic. In particular, if the rings are commutative, then they are isomorphic.*

Because of the above theorem, Morita equivalence is interesting solely in the situation of noncommutative rings.

Example 1.3.3. Let R be a ring and S be the ring of $n \times n$ matrices with entries in R. We show that R and S are Morita equivalent by finding an equivalence of categories $F\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$. We follow the proof of [Lam99, p. 470].

Let M, M', M'' be left R-modules, and let $f: M \to M'$ be a module homomorphism. Define the map $F\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ by

$$\begin{aligned} M &\mapsto M^n \\ F(f)\colon (m_1, \ldots, m_n)^\intercal &\mapsto (f(m_1), \ldots, f(m_n))^\intercal, \end{aligned}$$

where $m_1, \ldots, m_n \in M$ and $M^n = \{(m_1, m_2, \ldots, m_n)^\intercal : \forall i, m_i \in M\}$ is the module with the action defined by matrix multiplication from the left. We see that F is a functor since, for a module homomorphism $g\colon M' \to M''$, we have

$$\begin{aligned} F(g \circ f)(m_1, \ldots, m_n) &= (g \circ f(m_1), \ldots, g \circ f(m_n))^\intercal \\ &= F(g)(f(m_1), \ldots, f(m_n))^\intercal \\ &= F(g) \circ F(f)(m_1, \ldots, m_n)^\intercal, \end{aligned}$$

and one can easily check that $F(f)$ is a module homomorphism.

Let M, M' be left S-modules, and let $f: M \to M'$ be a module homomorphism. Define the map $G\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$ by

$$\begin{aligned} M &\mapsto E_{11}M \\ G(f)\colon E_{11}m &\mapsto E_{11}f(m), \end{aligned}$$

where $m \in M$ and E_{ij} is the matrix with 1 in the (i, j) position and 0 elsewhere. Letting r_{ij} be the matrix with $r \in R$ in the (i, j) position, we see that

$$rE_{11}M = r_{11}E_{11}M = E_{11}r_{11}M \subseteq E_{11}M,$$

so $E_{11}M$ is a left R-module with respect to multiplication.

If $g\colon M' \to M''$ is a module homomorphism, then

$$\begin{aligned} G(g \circ f)(E_{11}m) &= E_{11}g \circ f(m) \\ &= G(g)(E_{11}f(m)) \\ &= G(g) \circ G(f)(E_{11}m), \end{aligned}$$

and one can easily show $G(f)$ is a module homomorphism, so G is a functor.

Let M, M' be left R-modules. We want to show that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{{}_R\mathsf{Mod}}$. Let $\beta_M: M \to \{(m, 0, \ldots, 0) : m \in M\} = G \circ F(M)$ be the map defined by

$$\beta_M(m) = (m, 0 \ldots, 0).$$

Then β_M is an isomorphism and for $f: M \to M'$, we have

$$\begin{aligned}(G\circ F)(f)\circ\beta_M(m) &= (G\circ F)(f)(m,0,\dots,0)^\intercal\\ &= (G\circ F)(f)E_{11}(m,m,\dots,m)^\intercal\\ &= E_{11}(f(m),f(m),\dots,f(m))^\intercal\\ &= (f(m),0,\dots,0)^\intercal\\ &= \beta_{M'}\circ \mathrm{id}_{{}_R\mathsf{Mod}}(f)(m),\end{aligned}$$

so $\beta\colon \mathrm{id}_{{}_R\mathsf{Mod}} \to G\circ F$ is a natural isomorphism.

Let M be a left S-module. To show that F is an equivalence of categories, we must also show that $F\circ G \cong \mathrm{id}_{{}_S\mathsf{Mod}}$. We do this by finding a natural isomorphism $\alpha_M\colon M \to (E_{11}M)^n = F(G(M))$. Define the map α_M by

$$\alpha_M(m) = (E_{11}m, E_{12}m, \dots, E_{1n}m)^\intercal,$$

where $m \in M$. Notice that $\alpha_M(m+m')$ for $m' \in M$, so to show that α_M is a module homomorphism, we must demonstrate that, given $r\in R$, we have $\alpha_M(rE_{ij}m) = rE_{ij}\alpha_M(m)$, because $\{rE_{ij}\colon r\in R, 1\le i\le n, 1\le j\le n\}$ forms a spanning set for S. The left-hand side evaluates to

$$\alpha_M(rE_{ij}m) = (0,\dots,0,\underbrace{rE_{1j}m}_{j\text{-th position}},0,\dots,0)^\intercal,$$

and the right-hand side evaluates to

$$rE_{ij}\alpha(m) = rE_{ij}(E_{11}m,\dots,E_{1n}m)^\intercal = (0,\dots,0,\underbrace{rE_{1j}m}_{j\text{-th position}},0,\dots,0)^\intercal,$$

so the equality is satisfied.

To show that α_M is an isomorphism, we show that it is injective and surjective. First, let $\alpha_M(m) = 0$ for some $m\in M$. Then

$$\begin{aligned}(E_{11}m,\dots,E_{1n}m)^\intercal = (0,\dots,0)^\intercal &\implies E_{1j}m = 0 \text{ for all } j\\ &\implies 0 = \sum_j E_{j1}E_{1j}m = \sum_j E_{jj}m = I_n m = m,\end{aligned}$$

so α_M is injective. Now, let $p_1,\dots,p_n \in E_{11}M$; one can then pick $m_1,\dots,m_n \in M$ such that $p_i = E_{11}m_i$ for $1\le i\le n$. Then,

$$\begin{aligned}\alpha_M(p_i) &= (E_{11}p_i,\dots,E_{1n}p_i)^\intercal\\ &= (E_{11}E_{11}m_i, E_{12}E_{11}m_i,\dots,E_{1n}E_{11}m_i)^\intercal\\ &= (p_i,0,\dots,0)^\intercal,\end{aligned}$$

so

$$(0,\dots,0,\underbrace{\;p_i\;}_{i\text{-th position}},0,\dots,0)^\intercal = E_{i1}(p_i,0,\dots,0)^\intercal = E_{i1}\alpha_M(p_i) = \alpha_M(E_{i1}p_i),$$

and finally,

$$\alpha_M(E_{11}p_1 + E_{21}p_2 + \cdots + E_{n1}p_n) = \alpha_M(E_{11}p_1) + \cdots + \alpha_M(E_{n1}p_n) = (p_1, \dots, p_n)^\intercal;$$

this shows us that α_M is surjective, and thus an isomorphism.

Let M, M' be two left S-modules, with $m \in M$, and let $f : M \to M'$ be a module homomorphism. Then

$$\begin{aligned}
(F \circ G)(f) \circ \alpha_M(m) &= (F \circ G)(f)(E_{11}m, \dots, E_{1n}m) \\
&= (F \circ G)(f)(E_{11}E_{11}m, E_{11}E_{12}m, \dots, E_{11}E_{1n}m) \\
&= (E_{11}f(E_{11}m), \dots, E_{11}f(E_{1n}m)) \\
&= (E_{11}E_{11}f(m), \dots, E_{11}E_{1n}f(m)) \\
&= (E_{11}f(m), \dots, E_{1n}f(m)) \\
&= \alpha_{M'} \circ \mathrm{id}_{{}_S\mathsf{Mod}}(f)(m),
\end{aligned}$$

so α is a natural isomorphism $\mathrm{id}_{{}_S\mathsf{Mod}} \to F \circ G$.

We finally see that $F : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is an equivalence of categories, so R and S are Morita equivalent.

For every field $\mathbb{K}$ and finite-dimensional $\mathbb{K}$-algebra A, there is an associated basic algebra (see Definition 1.2.12) defined in the following way: let Ae be a minimal progenerator of the full subcategory of ${}_A\mathsf{Mod}$ consisting of finite-dimensional left A-modules, with $e = e^2 \in A$. Then $A^b = eAe$ is called the basic algebra associated to A. One can choose e in a particular way. We can decompose the identity

$$1_A = \sum_{j=1}^{s_1} e_{1j} + \cdots + \sum_{j=1}^{s_n} e_{nj} = \sum_{i=1}^{n} \sum_{j=1}^{s_j} e_{ij}$$

into the sum of pairwise primitive orthogonal idempotents e_{ij} (see Definition 1.2.14) such that

$$Ae_{ij} \cong Ae_{ik} \quad \text{for } j,k \in \{1, \dots, s_i\}, \quad Ae_{ij} \ncong Ae_{i'k} \quad \text{for } i \neq i'.$$

Then we may let $e = \sum_{n=1}^{n} e_{i1}$. In fact, every decomposition of A into a finite number of left ideals induces a partition $1_A = e_1 + \cdots + e_r$, as is shown by the following proposition, to which we present the proof due to an error in the book.

Proposition 1.3.4 ([Coh03, Proposition 4.3.1])**.** *Let R be any ring. Any decomposition of R as a direct sum of a finite number of left ideals*

$$R = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r \tag{1.1}$$

corresponds to a decomposition of 1 *as a sum of pairwise orthogonal idempotents*

$$1 = e_1 + \cdots + e_r, \tag{1.2}$$

where $\mathfrak{a}_i = Re_i$, and for any idempotent e, Re is indecomposable if and only if e is primitive.

Proof. Given (1.1), we write 1 as $\sum_i e_i$ where $e_i \in \mathfrak{a}_i$. Then $e_k = \sum_i e_k e_i$; since the sum (1.1) is direct, we see that $e_k e_i = 0$ of $i \neq k$ and $e_k^2 = e_k$, so (1.2) follows.

Conversely, given (1.2), put $\mathfrak{a}_i = Re_i$; then any $x \in R$ can be written as $x = \sum_i xe_i$, where $xe_i \in \mathfrak{a}_i$, hence $R = \sum_i \mathfrak{a}_i$ and this sum is direct, for if $\sum_i x_i e_i = 0$, then right multiplication by e_k gives $x_k e_k = 0$.

For any idempotent e, Re is a direct summand of R: $R = Re + R(1-e)$; now the above correspondence shows that any direct decomposition of Re corresponds to writing e as a sum of two orthogonal idempotents. It follows that Re is indecomposable if and only if e is primitive. □

Thus, if A decomposes into

$$A = Ae_{11} \oplus Ae_{12} \oplus \cdots \oplus Ae_{ns_n},$$

and e is the idempotent given in $A^b = eAe$, then Ae is obtained from A by removing all the duplicate indecomposable submodules, as these duplicates are killed by e.

$$Ae = (Ae_{11} \oplus Ae_{12} \oplus \cdots \oplus Ae_{ns_n})e = Ae_{11} \oplus Ae_{21} \oplus \cdots \oplus Ae_{n1}.$$

The ideas discussed in the above discussion are key to the proof of the following theorem.

Theorem 1.3.5 ([SY11, Theorem 6.16]). *Every finite-dimensional $\mathbb{K}$-algebra A is Morita equivalent to its basic algebra A^b.*

1.4 Morita Theory

We now seek to explore key theorems with regards to Morita equivalence, and their consequences, that Kiiti Morita found and proved in [Mor58]. Our exposition is heavily influenced by [Lam99].

Definition 1.4.1 (Trace ideal). The trace ideal of a left (right) R-module M is denoted $\mathrm{tr}(M)$ and is defined by

$$\mathrm{tr}(M) := \sum_{f \in \mathrm{Hom}_{R\mathsf{Mod}}(M,R)} f(M).$$

Definition 1.4.2 (Generator). For a ring R, a right R-module P is called a *generator* for Mod_R if $\mathrm{Hom}_R(P,-)\colon \mathsf{Mod}_R \to \mathsf{Ab}$ is a faithful functor. In otherwords, if for $M, N \in \mathrm{Ob}(\mathsf{Mod}_R)$ there is a morphism $f\colon M \to R$, then $\mathrm{Hom}_R(P, f)$ is nonzero.

The following theorem makes clear the many interesting properties of generators.

Theorem 1.4.3 ([Lam99, Theorem 18.8]). *For any right R-module P, the following are equivalent:*

1. *P is a generator;*

2. $\operatorname{tr}(P) = R$;

3. *R is a direct summand of a finite direct sum $\bigoplus_i P$;*

4. *R is a direct summand of a direct sum $\bigoplus_i P$; and*

5. *every $M \in \mathrm{Ob}(\mathsf{Mod}_R)$ is an surjective image of some direct sum $\bigoplus_i P$.*

Definition 1.4.4 (Projective module)**.** A left R-module P is said to be *projective* if, for any surjective homomorphism of left R-modules $g\colon B \to C$, and any R-homomorphism $h\colon P \to C$, there exists an R-homomorphism $h'\colon P \to B$ such that $h = g \circ h'$.

Definition 1.4.5 (Progenerator)**.** Given a ring R, a left R-module P is a *progenerator* if P a is finitely generated, projective generator.

Definition 1.4.6 (Full idempotent)**.** Given a ring R, an idempotent $x \in R$ of R is called a *full idempotent* of R if $RxR = R$. That is, the two-sided ideal generated by x is the whole ring R.

Example 1.4.7. An example of a full idempotent $e \in M_n(R)$ is $e = E_{kk}$ for $1 \leq k \leq n$. This is because $E_{ik}E_{kk}E_{kj} = E_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, and $\{E_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ is a basis for $M_n(R)$.

Definition 1.4.8 (Bimodule)**.** Given two rings R and S, an (R,S)-*bimodule* M is an abelian group such that,

- M is a left R-module and a right S-module; and
- for $r \in R, s \in S, m \in M$, we have $(rm)s = r(ms)$.

We may write ${}_RM_S$ to emphasize the (R,S)-bimodule structure.

Example 1.4.9. A ring R is itself an (R,R)-bimodule with respect to the ring multiplication.

Example 1.4.10. Given a ring R, the ring $M_{m\times n}(R)$ of $m \times n$ matrices with entries from R is an $(M_m(R), M_n(R))$-bimodule with respect to standard matrix multiplication.

Example 1.4.11. Given a commutative ring R, a left R-module M is also a right R-module. Let $r \in R$ and $m \in M$. One defines the right action by $mr = rm$; because R is a commutative ring, the axioms for a right module are satisfied. The left and right action also commute, so M is an (R,R)-bimodule.

Definition 1.4.12 (Bimodule homomorphism)**.** Given two (R,S)-bimodules M and N, a map $f: M \to N$ is a *bimodule homomorphism* if it is a homomorphism of left R-modules and a homomorphism of right S-modules.

We see in the next two propositions that tensor products interact in a useful way with bimodules.

Proposition 1.4.13. *If M is an (R,S)-bimodule and N is a left S-module, then $M \otimes_S N$ is a left R-module.*

Proof. Every element of $M \otimes_R N$ can be written as a finite sum $\sum_i m_i \otimes n_i$, so we define the left action to be

$$r\left(\sum_i m_i \otimes n_i\right) = \sum_i (rm_i) \otimes n_i.$$

We verify that this is an action.

$$\begin{aligned}
(r+r')\sum_i m_i \otimes n_i &= \sum_i (r+r')m_i \otimes n_i \\
&= \sum_i (rm_i + r'm_i) \otimes n_i \\
&= \sum_i rm_i \otimes n_i + \sum_i r'm_i \otimes n_i \\
(rr')\sum_i m_i \otimes n_i &= \sum_i (rr'm_i) \otimes n_i \\
&= \sum_i r(r'm_i) \otimes n_i \\
&= r\sum_i r'm_i \otimes n_i
\end{aligned}$$

This shows that $M \otimes_R N$ is a left R-module. □

Proposition 1.4.14. *If M is an (R,S)-bimodule and N is an (S,T)-bimodule, then $M \otimes_S N$ is an (R,T)-bimodule.*

Proof. Similar to the previous proof we see that $(\sum_i m_i \otimes n_i)t = \sum_i m_i \otimes (n_i t)$ defines a right action. It remains to show that the left and right action commute.

$$\begin{aligned}
r\left(\left(\sum_i m_i \otimes n_i\right)t\right) &= r\sum_i m_i \otimes n_i t \\
&= \sum_i rm_i \otimes n_i t \\
&= r\sum_i m_i \otimes n_i t \\
&= \left(r\sum_i m_i \otimes n_i\right)t
\end{aligned}$$

We conclude that $M \otimes_S N$ is an (R,T)-bimodule. □

Definition 1.4.15 (Endomorphism ring)**.** Given a left (right) R-module M, the *endomorphism ring*, denoted $\mathrm{End}_R(M)$ is the ring of all left (right) module homomorphisms of M into itself, with addition being pointwise addition, and multiplication being composition of homomorphisms.

Definition 1.4.16 (Faithfully balanced)**.** Given rings R and S, an (R,S)-bimodule M is called *faithfully balanced* if the ring homomorphisms $\lambda_M\colon R \to \mathrm{End}(M_S)$ and $\rho_M\colon S \to \mathrm{End}({}_RM)$ defined for $r \in R, s \in S, m \in M$ by

$$\lambda_M(r)(m) = r \cdot m \quad \text{and} \quad \rho_M(s)(m) = m \cdot s$$

are ring isomorphisms.

We follow closely the exposition of [Lam99, p. 485-490] up until Proposition 1.4.18.

Let R be a ring and P be a left R-module. Let $S = \mathrm{End}_R(P)$ be the endomorphism ring of P acting on the right of P by evaluation. That is, for $p \in P$ and $s \in S$, define

$$ps := s(p).$$

Note that this rule necessitates that composition is written in reverse. We verify that this is a right action: for $s, s' \in S$ and $p, p' \in P$,

$$\begin{aligned}(p+p')s &= s(p+p') = s(p) + s(p');\\ p(s+s') &= (s+s')(p) = s(p) + s'(p);\\ p(ss') &= p(s \circ s') = ((p)s)s' = (ps)s'\\ p1_S &= p.\end{aligned}$$

Thus P is a right S-module. The left and right actions on P commute because all elements of S are module homomorphisms. Thus P is an (R,S)-bimodule.

Let $Q = \mathrm{Hom}_R(P,R)$ be the set of left R-module homomorphisms $P \to R$ written on the right of P in the sense of $\mathrm{End}_R(P)$ above. Then, given the elements $q \in Q$, $p \in P$, and $r \in R$, Q is a right R-module with respect to the action $p(qr) = (pq)r$ ("PQR-associativity"). This is evidently a right action on Q as $pq \in R$ and q is a module homomorphism. Also, given the elements $q \in Q$, $p \in P$, and $s \in S$, Q is a left S-module with respect to the action $p(sq) = (ps)q$ ("PSQ-associativity"). Again, this is evidently a left action on Q as $ps \in P$ and q is a module homomorphism. The left and right actions on Q commute: given $p \in P$, $q \in Q$, $s \in S$, and $r \in R$,

$$(p(sq))r = ((ps)q)r = (ps)(qr) = p(s(qr)).$$

Thus, Q is an (S,R)-bimodule. Henceforth, p,q,r,s and p',q',r',s' will denote elements in P,Q,R, and S, respectively.

We now perform some calculations. Note that $pq \in R$ by the application of q to p (as noted above, Q is acting on the right of P) and $qp \in S$, which is defined as

$$(p')(qp) = (p'q)p \quad (\text{“}PQP\text{-associativity”}).$$

We verify that qp is an R-module homomorphism. Letting $p'' \in P$,

$$\begin{aligned}(p'+p'')(qp) &= ((p'+p'')q)p = (p'q+p''q)p = p'qp + p''qp;\\ rp'(qp) &= (rp)qp = r(p)qp.\end{aligned}$$

We now show that

$$(qp)q' = q(pq') \quad (\text{“}QPQ\text{-associativity”}).$$

This is checked by showing that the left and right sides are equal functions on P:

$$\begin{aligned}p'((qp)q') &= (p'(qp))q' && (\text{“}PSQ\text{-associativity”})\\ &= ((p'q)p)q' && (\text{“}PQP\text{-associativity”})\\ &= (p'q)(pq') && (\text{“}RPQ\text{-associativity”}: q \text{ is a left } R\text{-module homomorphism})\\ &= p'(q(pq')) && (\text{“}PQR\text{-associativity”}).\end{aligned}$$

Lemma 1.4.17 ([Lam99, Lemma 18.15])**.** *In the above notation:*

1. *$(p,q) \mapsto pq$ defines an (R,R)-bimodule homomorphism $\alpha\colon P \otimes_S Q \to R$;*
2. *$(q,p) \mapsto qp$ defines an (S,S)-bimodule homomorphism $\beta\colon Q \otimes_R P \to S$.*

Proof. We see that α is well defined due to PSQ-associativity. Further, α is an (R,R)-bimodule homomorphism due to RPQ-associativity and PQR-associativity. Similarly, β is well defined due to QRP-associativity, and is a bimodule homomorphism due to SQP-associativity and QPS-associativity. □

In the above context, the six-tuple $(R,P,Q,S;\alpha,\beta)$ is called the *Morita Context* associated to P. This Morita Context is fixed for the next two propositions and the following corollary.

Proposition 1.4.18 ([Lam99, Proposition 18.17])**.**

1. *The left R-module module P is a generator if and only if α is surjective.*
2. *Assume P is a generator. Then*
   a) *α is an isomorphism of (R,R)-bimodules;*
   b) *$Q \cong \mathrm{Hom}_S(P,S)$ as (S,R)-bimodules;*
   c) *$P \cong \mathrm{Hom}_S(Q,S)$ as (R,S)-bimodules; and*
   d) *$R \cong \mathrm{End}_S(P_S) \cong \mathrm{End}_S({}_SQ)$ as rings.*

Proposition 1.4.19 ([Lam99, Proposition 18.19])**.**

1. *The left R-module P is finitely generated and projective if and only if β is surjective.*
2. *Assume P is finitely generated projective. Then*
   a) *β is an isomorphism of (S,S)-bimodules;*
   b) *$Q \cong \mathrm{Hom}_R(P,R)$ as (R,S)-bimodules;*

c) $P \cong \mathrm{Hom}_R(S,R)$ as (S,R)-bimodules; and

d) $S \cong \mathrm{End}_R({}_RP) \cong \mathrm{End}_R(Q_R)$ as rings.

Corollary 1.4.20 ([Lam99, Corollary 18.21]). *If ${}_RP$ is a progenerator (i.e. a finitely generated projective generator), then ${}_RP_S$ and ${}_SQ_R$ are faithfully balanced bimodules.*

The next three theorems are those of [Mor58], and they help us see exactly when two rings are Morita equivalent.

Theorem 1.4.21 ([Lam99, Theorem 18.24] "Morita I"). *Let ${}_RP$ be a progenerator, and $(R,P,Q,S;\alpha,\beta)$ the associated Morita context. Then*

1. *$-\otimes_S Q\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ and $-\otimes_R P\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ are mutually inverse category equivalences.*
2. *$P\otimes_S -\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$ and $Q\otimes_R -\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ are mutually inverse category equivalences.*

Theorem 1.4.22 ([Lam99, Theorem 18.26] "Morita II"). *Let R and S be two rings, and*

$$F\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}, \quad G\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$$

be mutually inverse category equivalences. Let $Q = F({}_RR)$ and $P = G({}_SS)$. Then P is an (R,S)-bimodule and Q is an (S,R)-bimodule such that $F \cong Q\otimes_R -$ and $G = P\otimes_S -$.

Definition 1.4.23 ((S,R)-progenerator). Given two rings R and S, an (S,R)-bimodule P is called an *(S,R)-progenerator* if ${}_SP_R$ is faithfully balanced and P_R is a progenerator.

Theorem 1.4.24 ([Lam99, Theorem 18.28] "Morita III"). *Given two rings R and S, the isomorphism classes of category equivalences ${}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$ are in bijective correspondence with the isomorphism classes of (S,R)-progenerators. Composition of category equivalences corresponds to tensor products of such progenerators.*

The most significant consequences of Morita's theorems are the following.

Proposition 1.4.25 ([Lam99, Proposition 18.32]). *If ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ are equivalent, then Mod_R and Mod_S are equivalent.*

Proposition 1.4.26 ([Lam99, Proposition 18.33]). *Given rings R and S, the following are equivalent:*

1. *R is Morita equivalent to S.*
2. *$R \cong \mathrm{End}({}_SP)$ for some progenerator P of ${}_S\mathsf{Mod}$.*
3. *$R \cong eM_n(S)e$ for some full idempotent e in a matrix ring $M_n(S)$.*

We give some justification here for Theorem 1.3.2.

Proposition 1.4.27 ([Lam99, Lemma 18.41])**.** *Let M be a faithfully balanced (R,S)-bimodule. It follows that $Z(R) \cong Z(S)$, and R and S are isomorphic to the endomorphism ring of bimodule homomorphisms of M.*

Corollary 1.4.28 ([Lam99, Corollary 18.42])**.** *If R and S are Morita equivalent rings, then $Z(R) \cong Z(S)$ as rings.*

Proof. Since R and S are Morita equivalent, by Proposition 1.4.26, we know that $S \cong \mathrm{End}({}_R P)$ for ${}_R P$ a progenerator. Thus the Morita Context $(R, P, Q, S; \alpha, \beta)$ associated with P exists. We know by Corollary 1.4.20 that ${}_R P_S$ is a faithfully balanced (R,S)-bimodule. By applying Proposition 1.4.27 to P, we see that $Z(R) \cong Z(S)$. □

1.5 Bicategory of Bimodules

Here, we introduce a new tool with which to look at the relationship between rings, bimodules between them, and the homomorphisms between these bimodules: the bicategory. We begin discussing the idea of a 2-category, as this is helps us in dealing with bicategories.

Definition 1.5.1 (2-category)**.** A 2-*category* $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$ of objects or 0-*cells*;
- for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a category $\mathrm{Hom}_{\mathcal{C}}(A,B)$ whose objects $f, g, \ldots : A \to B$ are called 1-*morphisms* or 1-*cells*, whose morphisms $\alpha, \beta, \ldots$ are called 2-*morphisms* or 2-*cells*, and whose composition is denoted $\circ_1$ and called vertical composition;

$$\begin{tikzcd} A \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, "g" description, ""{name=G}] \arrow[r, bend right=60, "h"', ""{name=H, above}] & B \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, bend right=60, "h"', ""{name=H, above}] & B \arrow[Rightarrow, from=F, to=H, "\beta\circ_1\alpha"] \end{tikzcd}$$

- for every three objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ an associative, unital functor

$$\circ_0 \colon \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C),$$

called *horizontal composition* with identities id_A and $\mathrm{id}_{\mathrm{id}_A}$.

$$\begin{tikzcd} A \arrow[r, bend left=50, "f", ""{name=F, below}] \arrow[r, bend right=50, "g"', ""{name=G, above}] & B \arrow[r, bend left=50, "f'", ""{name=F2, below}] \arrow[r, bend right=50, "g'"', ""{name=G2, above}] & C \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=F2, to=G2, "\alpha'"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left=60, "f'\circ f", ""{name=F, below}] \arrow[r, bend right=60, "g'\circ g"', ""{name=G, above}] & C \arrow[Rightarrow, from=F, to=G, "\alpha'\circ_0\alpha"] \end{tikzcd}$$

Definition 1.5.2 (2-functor)**.** Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, a 2-*functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a consists of:

- a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$;
- given two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a function
$$F\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}'}(F(A),F(B));$$
- given two 1-morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, a function
$$\mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}(f,g) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(F(A),F(B))}(F(f),F(g))$$

which much satisfy the following:

- given an object $A \in \mathrm{Ob}(\mathcal{C})$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$;
- given objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f\colon A \to B$ and $g\colon B \to C$, we have $F(g \circ f) = F(g) \circ F(f)$;
- given objects $A, B \in \mathrm{Ob}(\mathcal{C})$ and a morphism $f\colon A \to B$, we have $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$;
- given 1-morphisms $f, g, h \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, and 2-morphisms $\alpha\colon f \to g$ and $\beta\colon g \to h$,
$$F(\beta \circ_1 \alpha) = F(\beta) \circ_1 F(\alpha);$$
and
- given 1-morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X,Y), f', g' \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ and 2-morphisms $\alpha\colon f \to g, \alpha'\colon f' \to g'$,
$$F(\alpha' \circ_0 \alpha) = F(\alpha') \circ_0 F(\alpha).$$

Example 1.5.3 (2-category of small categories)**.** There exists a 2-category Cat whose objects are small categories, whose 1-morphisms are functors, and whose 2-morphisms are natural transformations.

Example 1.5.4 (2-category of topological spaces)**.** There exists a 2-category Top whose objects are topological spaces, whose 1-morphisms are continuous maps, and whose 2-morphisms are homotopy classes.

Definition 1.5.5 (Bicategory)**.** A *bicategory* $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$ of objects or 0-*cells*;
- for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a category $\mathrm{Hom}_{\mathcal{C}}(A,B)$ whose objects $f, g, \ldots\colon A \to B$ are called 1-*morphisms* or 1-*cells*, whose morphisms $\alpha, \beta, \ldots$ are called 2-*morphisms* or 2-*cells*, and whose composition is denoted $\circ_1$ and called vertical composition;

- for every three objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a functor

$$\circ_0 \colon \operatorname{Hom}_{\mathcal{C}}(B,C) \times \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{C}}(A,C),$$

  called *horizontal composition*, such that for every five objects A, B, C, D, E and every four 1-morphisms $f \colon A \to B, g \colon B \to C, h \colon C \to D, k \colon D \to E$, there are 2-isomorphisms

$$\begin{aligned} \alpha_{f,g,h} &\colon h \circ (g \circ f) \to (h \circ g) \circ f \\ \lambda_f &\colon \mathrm{id}_B \circ f \to f \\ \rho_f &\colon f \circ \mathrm{id}_A \to f \end{aligned}$$

  such that the following diagrams commute.

$$\begin{tikzcd}
 & ((kh)g)f \arrow[rr, "\alpha\circ_0 1"] \arrow[dl, "\alpha"] & & ((kh)g)f \arrow[dr, "\alpha"] & \\
(kh)(gf) \arrow[drr, "\alpha"] & & & & k((hg)f) \arrow[dll, "1\circ_0\alpha"] \\
 & & k(h(gf)) & &
\end{tikzcd}$$

$$\begin{tikzcd}
(g\,\mathrm{id}_B)f \arrow[rr, "\alpha"] \arrow[dr, "\rho\circ_0 1"'] & & g(\mathrm{id}_B f) \arrow[dl, "1\circ_0\lambda"] \\
 & gf &
\end{tikzcd}$$

Example 1.5.6 (Bicategory of bimodules)**.** There exists a bicategory Bim whose objects are rings, whose 1-morphisms from R to S are (S,R)-bimodules, and whose 2-morphisms are bimodule homomorphisms. The composition

$$\operatorname{Hom}_{\mathsf{Bim}}(S,T) \times \operatorname{Hom}_{\mathsf{Bim}}(R,S) \to \operatorname{Hom}_{\mathsf{Bim}}(R,T)$$

is the tensor product over S.

This is presented as a bicategory and not a 2-category because, for an (R,S)-bimodule M, an (S,T)-bimodule N, and a (T,V)-bimodule P, there is an isomorphism

$$M \otimes_S (N \otimes_T P) \cong (M \otimes_S N) \otimes_T P,$$

however this does not constitute an equality. An isomorphism can be constructed using the universal property of tensor products.

Theorem 1.5.7 ([Hon, Theorem 3.2])**.** *Every bicategory is equivalent to a 2-category.*

Remark 1.5.8. In light Theorem 1.5.7, one can treat Bim as a 2-category by picking and using a 2-category equivalent to Bim.

For any given bimodule, there is a tensor product functor. Specifically, letting R, S be rings, and M an (R, S)-bimodule, we can define $M \otimes_S -\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$ for any left S-module N and any left module homomorphism f defined on N by

$$N \mapsto M \otimes_S N$$
$$M \otimes_S (f)\colon \sum_i m_i \otimes n_i \mapsto \sum_i m_i \otimes f(n_i),$$

and this is a functor because, for any additional left module homomorphism g defined on N,

$$\begin{aligned} M \otimes_S (g \circ f)\left(\sum_i m_i \otimes n_i\right) &= \sum_i m_i \otimes ((g \circ f)(n_i)) \\ &= (M \otimes_S g)\left(\sum_i m_i \otimes f(n_i)\right) \\ &= (M \otimes_S g)(M \otimes_S f)\left(\sum_i m_i \otimes n_i\right). \end{aligned}$$

Similarly, every bimodule homomorphism gives rise to a natural transformation between a pair of the aforementioned tensor product functors. In particular, for (R, S)-bimodules M and N, and a bimodule homomorphism $f\colon M \to N$, we get a natural transformation $\alpha\colon M \otimes_S - \to N \otimes_S -$ defined, for any left S-module A, $a_i \in A$, and $m_i \in M$, by

$$\alpha_A\left(\sum_i m_i \otimes a_i\right) = \sum_i f(m_i) \otimes a_i.$$

This is natural because, for any left module homomorphism $g\colon A \to B$ for left S-modules A, B, we have

$$\begin{aligned} N \otimes_S (g) \circ \alpha_A\left(\sum_i m_i \otimes a_i\right) &= (N \otimes_S (g))\left(\sum_i f(m_i) \otimes a_i\right) \\ &= \sum_i f(m_i) \otimes g(a_i) \\ &= \alpha_B\left(\sum_i m_i \otimes g(a_i)\right) \\ &= \alpha_B \circ (M \otimes_S (g))\left(\sum_i m_i \otimes a_i\right). \end{aligned}$$

This gives a 2-functor $F\colon \mathsf{Bim} \to \mathsf{Cat}$ defined by

$$\begin{aligned} R &\mapsto {}_R\mathsf{Mod}, \\ {}_RM_S &\mapsto {}_RM_S \otimes_S -, \text{ and} \\ ({}_RM_S \to {}_RN_S) &\mapsto (M \otimes_S - \to N \otimes_S -). \end{aligned}$$

To see that F is a 2-functor, we first note that, for two composable bimodules ${}_RM_S$ and ${}_SN_T$,

$$\begin{aligned} F({}_TM_S \circ {}_SN_R) &= F({}_TM_S \otimes_S {}_SN_R) \\ &= {}_TM_S \otimes_S {}_SN_R \otimes_R - \\ &= ({}_TM_S \otimes_S -) \circ ({}_SN_R \otimes_R -) \\ &= F({}_TM_S) \circ F({}_SN_R). \end{aligned}$$

Next, for two vertically composable bimodule homomorphisms $f\colon {}_RM_S \to {}_RN_S$ and $g\colon {}_RN_S \to {}_RP_S$, and for a left S-module A,

$$\begin{aligned} F(g \circ_1 f)_A\left(\sum_i m_i \otimes a_i\right) &= \sum_i (g \circ_1 f)(m_i) \otimes a_i \\ &= F(g)_A\left(\sum_i f(m_i) \otimes a_i\right) \\ &= (F(g)_A \circ_1 F(f)_A)\left(\sum_i m_i \otimes a_i\right). \end{aligned}$$

Finally, for two horizontally composable bimodule homomorphisms $f\colon {}_SN_T \to {}_SN'_T$ and $g\colon {}_RM_S \to {}_RM'_S$, and left T-module A,

$$\begin{aligned} F(g \circ_0 f)_A\left(\sum_i m_i \otimes n_i \otimes a_i\right) &= \left(\sum_i g(m_i) \otimes f(n_i) \otimes a_i\right) \\ &= (F(g) \circ_0 F(f))\left(\sum_i m_i \otimes n_i \otimes a_i\right). \end{aligned}$$

1.6 Morita Theory Revisited

We now look for ways of applying knowledge of bicategories, and in particular Bim, towards an understanding of Morita theory. We begin by addressing how Bim nicely deals with the concept of Morita equivalence.

Definition 1.6.1 (Isomorphism of objects)**.** Given a category $\mathcal{C}$ and two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, an *isomorphism* $f\colon A \to B$ is a morphism which is both right-invertible and left-invertible.

Definition 1.6.2 (Equivalence of objects)**.** Given a bicategory $\mathcal{C}$, two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ are called *equivalent* if there exist morphisms $f: A \to B$ and $g: B \to A$ such that $fg \cong \mathrm{id}_B$ and $gf \cong \mathrm{id}_A$. The maps f and g satisfying this property are called *equivalences*.

Theorem 1.6.3. *The rings R and S are Morita equivalent if and only if R and S are equivalent in* Bim.

Proof. By Corollary 1.6.18, we know that if R and S are Morita equivalent, then there are 1-morphisms ${}_SQ_R$ and ${}_RP_S$ in Bim such that when horizontally composed,

$$ {}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R \cong 1_{{}_R\mathsf{Mod}} \quad \text{and} \quad {}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S \cong 1_{{}_S\mathsf{Mod}}, $$

showing that ${}_RP_S$ is an isomorphism $S \to R$ in Bim. By again applying Corollary 1.6.18, the converse holds by inspection. □

Now, we move towards giving a proof of Corollary 1.6.18 using a bicategorical perspective. First, we must understand exact sequences so that we can deal with Theorem 1.6.17.

Definition 1.6.4 (Exact sequence)**.** A sequence of left R-modules and module homomorphisms

$$ \cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \cdots $$

is called an *exact sequence* if $\operatorname{im} f_{n+1} = \ker f_n$ for all n.

Definition 1.6.5 (Short exact sequence)**.** A sequence of the form

$$ 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, $$

where 0 is the trivial module, is called a *short exact sequence*

Definition 1.6.6 (Left exact functor)**.** A functor $T: {}_R\mathsf{Mod} \to \mathsf{Ab}$ is called *left exact* if exactness of

$$ 0 \to A \xrightarrow{f} B \xrightarrow{g} C $$

implies exactness of

$$ 0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C). $$

Definition 1.6.7 (Right exact functor)**.** A functor $T: {}_R\mathsf{Mod} \to \mathsf{Ab}$ is called *right exact* if exactness of

$$ A \xrightarrow{f} B \xrightarrow{g} C \to 0 $$

implies exactness of

$$ T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \to 0. $$

Lemma 1.6.8 ([Rot09, Theorem 2.63]). *Given a right R-module M, the functor*

$$M \otimes_R - \colon {}_R\mathsf{Mod} \to \mathsf{Ab}$$

is right exact.

It is also useful to know how to deal with direct sums in module categories. This is where the notions of preadditive categories and additive functors can be used.

Definition 1.6.9 (Preadditive category). A category $\mathcal{C}$ is a *preadditive category* if,

- for every $A, B \in \mathrm{Ob}(\mathcal{C})$, there is a binary operation

  $$+_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,B),$$

  referred to simply as $+$;

- for every $A, B \in \mathrm{Ob}(\mathcal{C})$, $(\mathrm{Hom}_{\mathcal{C}}(A,B), +)$ is an abelian group with identity $0_{A,B}$;

- for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, $f, g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, we have $h(f+g) = hf + hg$;

- and for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, $f, g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(C,A)$, we have $(f+g)h = fh + gh$.

Definition 1.6.10 (Biproduct). Given a preadditive category $\mathcal{C}$, a *biproduct* for objects $A, B \in \mathrm{Ob}(\mathcal{C})$ is a diagram

$$\begin{tikzcd} A \arrow[r, bend right, "i_1"'] & C \arrow[l, bend right, "p_1"'] \arrow[r, bend right, "p_2"'] & B \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

whose morphisms p_1, p_2, i_1, i_2 satisfy

$$p_1 i_1 = 1_A, \quad p_2 i_2 = 1_B, \quad i_1 p_1 + i_2 p_2 = 1_C.$$

Example 1.6.11. Given a ring R, the category of left R-modules ${}_R\mathsf{Mod}$ is a preadditive category: addition of module homomorphisms is pointwise addition and the finite direct sum of modules is a biproduct.

Definition 1.6.12 (Additive functor). Given preadditive categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is an *additive functor* if for every $A, B \in \mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ is an abelian group homomorphism.

Proposition 1.6.13 ([Wei94, Theorem 2.6.1]). *If $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ are additive functors that form an adjoint pair (F, G), then F is right exact and G is left exact.*

Lemma 1.6.14 ([Rot09, p. 74]). *Given a right R-module M and a left R-module N, the functors*

$$M \otimes_R - \colon {}_R\mathsf{Mod} \to \mathsf{Ab} \quad \textit{and} \quad - \otimes_R N \colon \mathsf{Mod}_R \to \mathsf{Ab}$$

are additive functors.

Proposition 1.6.15 ([ML98, p. 197]). *Given preadditive categories $\mathcal{C}$ and $\mathcal{D}$ where $\mathcal{C}$ has all binary biproducts, then a functor $T\colon \mathcal{C} \to \mathcal{D}$ commutes with biproducts if and only if T is an additive functor.*

According to [Mey, p. 1], Theorem 1.6.17 was simultaneously found by Eilenberg, Gabriel, and Watts. We follow Watts's proof. But first, we need a lemma.

Lemma 1.6.16. *For a left R-module M, $M \cong {}_RR_R \otimes_R M$ as left R-modules.*

Proof. Consider the map $f\colon {}_RR_R \otimes_R M \to M$ defined by $f(\sum_i r_i \otimes m_i) = \sum_i r_i m_i$ for $r_i \in R$ and $m_i \in M$. As

$$\begin{aligned} f\left(\sum_{i=0}^{a} r_i \otimes m_i + \sum_{i=a+1}^{b} r_i \otimes m_i\right) &= f\left(\sum_{i=0}^{a+b} r_i \otimes m_i\right) \\ &= \sum_{i=0}^{a+b} r_i m_i \\ &= \sum_{i=0}^{a} r_i m_i + \sum_{i=a+1}^{b} r_i m_i \\ &= f\left(\sum_{i=0}^{a} r_i \otimes m_i\right) + f\left(\sum_{i=0}^{b} r_i \otimes m_i\right), \end{aligned}$$

and

$$\begin{aligned} f\left(r \sum_{i} r_i \otimes m_i\right) &= f\left(\sum_{i} r r_i \otimes m_i\right) \\ &= \sum_{i} r r_i m_i \\ &= r \sum_{i} r_i m_i \\ &= r f\left(\sum_{i} r_i \otimes m_i\right), \end{aligned}$$

we see that f is a module homomorphism. It is evident that f is surjective as, for any $m \in M$, $f(1_R \otimes m) = m$. Finally, f is injective, as

$$\begin{aligned} f\left(\sum_{i} r_i \otimes m_i\right) = 0 &\implies \sum_{i} r_i m_i = 0 \\ &\implies \sum_{i} 1 \otimes r_i m_i = 0 \\ &\implies \sum_{i} r_i \otimes m_i = 0. \end{aligned}$$

Together, this shows that f is an isomorphism of left R-modules. □

Theorem 1.6.17 ([Wat60, Theorem 1])**.** *Let $T\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ be a right-exact functor that commutes with direct sums. Then there is an (S,R)-bimodule Q and a natural equivalence of functors $\psi\colon Q\otimes_R - \to T$.*

Proof. Given $M \in \mathrm{Ob}({}_R\mathsf{Mod})$ and $m \in M$, define $\phi_m\colon R \to M$ by $\phi_m(r) = rm$. We define a function $\psi_0^M\colon TR \times M \to TM$ by $\psi_0^M(\bar{r}, m) = (T\phi_m)(\bar{r})$ for $\bar{r} \in TR$ and $m \in M$. Note that T is additive by Proposition 1.6.15. Further, $\psi_0^R\colon TR \times R \to TR$ defines a right action, because for $\bar{r} \in TR$, $r_1, r_2 \in R$,

$$\begin{aligned}
\psi_0^R(\psi_0^R(\bar{r}, r_1), r_2) &= \psi_0^R((T\phi_{r_1})(\bar{r}), r_2)\\
&= (T\phi_{r_2})((T\phi_{r_1})(\bar{r}))\\
&= T(\phi_{r_2} \circ \phi_{r_1})(\bar{r}) \quad \text{by functoriality of } T\\
&= T(\phi_{r_1 r_2})(\bar{r})\\
&= \psi_0^R(\bar{r}, r_1 r_2);
\end{aligned}$$

$$\begin{aligned}
\psi_0^R(\bar{r}, r_1 + r_2) &= (T\phi_{r_1+r_2})(\bar{r})\\
&= T(\phi_{r_1} + \phi_{r_2})(\bar{r})\\
&= (T\phi_{r_1})(\bar{r}) + (T\phi_{r_2})(\bar{r}) \quad \text{by additivity of } T\\
&= \psi_0^R(\bar{r}, r_1) + \psi_0^R(\bar{r}, r_2);
\end{aligned}$$

for $\bar{r_1}, \bar{r_2} \in TR, r \in R$,

$$\begin{aligned}
\psi_0^R(\bar{r_1} + \bar{r_2}, r) &= T(\phi_r)(\bar{r_1} + \bar{r_2})\\
&= T(\phi_r)(\bar{r_1}) + T(\phi_r)(\bar{r_2}) \quad \text{since } T\phi_r \text{ is a module homomorphism}\\
&= \psi_0^R(\bar{r_1}, r) + \psi_0^R(\bar{r_2}, r);
\end{aligned}$$

and for $1_R \in R, \bar{r} \in TR$,

$$\psi_0^R(\bar{r}, 1_R) = (T\phi_{1_R})(\bar{r}) = (T1_R)(\bar{r}) = 1_{TR}(\bar{r}) = \bar{r}.$$

Finally, the left and right actions on TR commute: for $s \in S$, $r \in R$, $\bar{r} \in TR$,

$$s\psi_0^R(\bar{r}, r) = s(T\phi_r)(\bar{r}) = (T\phi_r)(s\bar{r}) = \psi_0^R(s\bar{r}, r).$$

Thus TR is an (S,R)-bimodule. TR will henceforth be denoted by Q.

Note that ψ_0^M, for $M \in \mathrm{Ob}({}_R\mathsf{Mod})$ is R-balanced because it is defined by an action. Thus, ψ_0^M descends to an R-homomorphism $\psi^M\colon Q\otimes_R M \to TM$, and the family $(\psi^M)_{M\in\mathrm{Ob}({}_R\mathsf{Mod})}$ is a natural transformation $\psi\colon Q\otimes_R - \to T$. For any left R-module homomorphism $f\colon M \to M'$,

$$\begin{aligned}
T(f) \circ \psi^M(x \otimes m) &= T(f)(T\phi_m(x))\\
&= T(f \circ \phi_m)(x)\\
&= T(\phi_{f(m)})(x)\\
&= \psi^{M'}(x \otimes f(m))\\
&= \psi^{M'} \circ Q\otimes_R (f)(x \otimes m).
\end{aligned}$$

Further, ψ^R is a natural isomorphism, as $C \otimes_R R = TR \otimes_R R \cong TR$ by Lemma 1.6.16. T commutes with direct sums, and so does $Q \otimes_R -$ by Lemma 1.6.14, so ψ^F is an isomorphism whenever F is a free left R-module, as F is the direct sum of copies of R.

Let $M \in \mathrm{Ob}({}_R\mathsf{Mod})$. By Proposition 1.2.20, we can obtain a free left R-module F admitting a short exact sequence:

$$0 \to L \to F \to M \to 0$$

Seeing that T and $Q \otimes_R -$ are right exact (the latter by Lemma 1.6.8), the following diagram of exact rows commutes.

$$\begin{tikzcd}
Q \otimes_R L \arrow[r, "\alpha_1"] \arrow[d, "\psi^L"] & Q \otimes_R F \arrow[r, "\beta_1"] \arrow[d, "\psi^F"] & Q \otimes_R M \arrow[r] \arrow[d, "\psi^M"] & 0 \\
TL \arrow[r, "\alpha_2"] & TF \arrow[r, "\beta_2"] & TA \arrow[r] & 0
\end{tikzcd}$$

Note that ψ^L and ψ^M are surjective because they are right actions on TL and TM, respectively. To show that ψ^M is injective, we "chase" the diagram.

Let $a \in \ker \psi^M$. Since β_1 is surjective, there is $b \in Q \otimes_R F$ such that $\beta_1(b) = a$. Let $c = \psi^F(b) \in TF$. Then

$$\beta_2(c) = \beta_2(\psi^F(b)) = \psi^M(\beta_1(b)) = \psi^M(a) = 0.$$

Since $c \in \ker \beta_2$, we know that $c \in \operatorname{im} \alpha_2$, so we can pick $d \in TL$ such that $\alpha_2(d) = c$. Since ψ^L is surjective, we can pick $e \in Q \otimes_R L$ such that $\psi^L(e) = d$. Since ψ^F is injective, b is the unique preimage of c through ψ^F. Since the diagram commutes, we know that $\alpha_1(e) = b$. Finally, by exactness,

$$a = \beta_1(b) = \beta_1(\alpha_1(e)) = 0.$$

Thus, ψ^M is injective, and hence an isomorphism. □

We can now use our knowledge of Bim to prove the following corollary.

Corollary 1.6.18 ([Mey, Corollary 1.1]). *The rings R and S are Morita equivalent if and only if there is an (R,S)-bimodule P and an (S,R)-bimodule Q such that ${}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R$ and ${}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S$ as bimodules.*

Proof. ($\Longleftarrow$) Using the 2-functor F defined previously, we have that

$${}_RP_S \otimes_S {}_SQ_R \otimes_R - = F({}_RP_S \otimes_S {}_SQ_R) \cong F({}_RR_R) = {}_RR_R \otimes_R -,$$

so ${}_RP_S \otimes_S {}_SQ_R \otimes_R - \cong 1_{{}_R\mathsf{Mod}}$ by Lemma 1.6.16. Similarly, ${}_SQ_R \otimes_R {}_RP_S \otimes_S - \cong 1_{{}_S\mathsf{Mod}}$. Thus ${}_RP_S \otimes_S -$ is an equivalence of categories ${}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$.

($\Longrightarrow$) Let $T\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ be an equivalence of categories. It is well-known that equivalences are additive, and so T is additive. Since T is additive and the left adjoint to its inverse, we see that T commutes with direct sums and is right exact by Proposition 1.6.13, so T satisfies the conditions of Theorem 1.6.17. Thus there is an (S,R)-bimodule Q such that $Q \otimes_R -$ is equivalent to T. Similarly, for the inverse equivalence $U\colon {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$, there is an (R,S)-bimodule P such that $P \otimes_S -$ is equivalent to U. Thus

$$F({}_RR_R) = {}_RR_R \otimes_R - \cong 1_{{}_R\mathsf{Mod}} \cong {}_RP_S \otimes_S {}_SQ_R \otimes_R - = F({}_RP_S \otimes_S {}_SQ_R),$$

so ${}_RR_R \cong {}_RP_S \otimes_S {}_SQ_R$, and similarly ${}_SS_S \cong {}_SQ_R \otimes_R {}_RP_S$. □

We now give an example of how Corollary 1.6.18 can be applied to determining a Morita equivalence.

Example 1.6.19. Let R be a ring and consider its $n \times n$ matrix ring $M_n(R)$. Then the set of $n \times 1$ matrices $M_{n\times 1}(R)$ is a left $M_n(R)$-module with respect to matrix multiplication and a right R-module with respect to scalar multiplication, and both actions commute, so $M_{n\times 1}(R)$ is a $(M_n(R), R)$-bimodule. Similarly, $M_{1\times n}(R)$ is an $(R, M_n(R))$-bimodule.

We check that $f\colon M_{1\times n}(R) \times M_{n\times 1}(R) \to R$ defined by

$$f((r_1,\dots,r_n),(s_1,\dots,s_n)^\intercal) = \sum_{i=1}^{n} r_i s_i$$

is $M_n(R)$-balanced and $g\colon M_{n\times 1}(R) \times M_{1\times n}(R) \to M_n(R)$ defined by

$$g((r_1,\dots,r_n)^\intercal,(s_1,\dots,s_n)) = (r_i s_j)$$

is R-balanced. However, it is evident that both of these maps are given by matrix multiplication, and since matrix multiplication is associative and distributive, we have that f and g are $M_n(R)$-balanced and R-balanced, respectively. Thus f and g both descend to well-defined bimodule homomorphisms $\tilde{f}\colon M_{1\times n}(R) \otimes_{M_n(R)} M_{n\times 1} \to {}_RR_R$ and $\tilde{g}\colon M_{n\times 1} \otimes_R M_{1\times n}(R) \to_{M_n(R)} M_n(R)_{M_n(R)}$ defined by

$$\tilde{f}((r_1,\dots,r_n)\otimes(s_1,\dots,s_n)^\intercal) = \sum_{i=1}^{n} r_i s_i$$

and

$$\tilde{g}((r_1,\dots,r_n)^\intercal \otimes (s_1,\dots,s_n)) = (r_i s_j).$$

We check that $\tilde{f}$ and $\tilde{g}$ are bimodule isomorphisms. First, $\tilde{f}$ is surjective: for any $r \in R$,

$$\tilde{f}((r,1,\dots,1)\otimes(1,\dots,1)^\intercal) = r.$$

Note that every simple tensor $(r_1,\dots,r_n)\otimes(s_1,\dots,s_n)^\intercal$ can be rewritten in a particular way:

$$
\begin{aligned}
(r_1,\dots,r_n)\otimes(s_1\dots s_n)^\intercal &= (r_1,\dots,r_n)\otimes\begin{pmatrix} s_1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ s_n & 0 & 0 & 0\end{pmatrix}(1,0,\dots,0)^\intercal\\
&= (r_1,\dots,r_n)\begin{pmatrix} s_1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ s_n & 0 & 0 & 0\end{pmatrix}\otimes(1,0,\dots,0)^\intercal\\
&= \left(\sum_i r_i s_i, 0,\dots,0\right)\otimes(1,0\dots,0)^\intercal
\end{aligned}
$$

The map is thus also injective:

$$
\begin{aligned}
\tilde{f}((r_1,\dots,r_n)\otimes(s_1,\dots,s_n)^\intercal) = 0 &\implies \sum_{i=1}^n r_i s_i = 0\\
&\implies \left(\sum_i r_i s_i, 0,\dots,0\right)\otimes(1,0,\dots,0)^\intercal = 0\\
&\implies (r_1,\dots,r_n)\otimes(s_1,\dots,s_n)^\intercal = 0.
\end{aligned}
$$

Since $\tilde{f}$ is R-balanced, it is a bimodule homomorphism, and hence a bimodule isomorphism. Now, $\tilde{g}$ is surjective:

$$
\tilde{g}((0,\dots,0,\underbrace{1}_{i\text{-th position}},0,\dots,0)^\intercal\otimes(0,\dots,0,\underbrace{1}_{j\text{-th position}},0,\dots,0)) = E_{ij},
$$

and $\{E_{ij} \mid 1\le i\le n, 1\le j\le n\}$ forms a basis for $M_n(R)$. Note that every simple tensor $(r_1,\dots,r_n)\otimes(s_1,\dots,s_n)^\intercal$ can be rewritten in a particular way:

$$
\begin{aligned}
(r_1,\dots,r_n)^\intercal\otimes(s_1,\dots,s_n) &= ((1,0,\dots,0)^\intercal r_1+\cdots+(0,\dots,0,1)^\intercal r_n)\otimes(s_1,\dots,s_n)\\
&= (1,0,\dots,0)^\intercal r_1\otimes(s_1,\dots,s_n)+\cdots+(0,\dots,0,1)^\intercal r_n\otimes(s_1,\dots,s_n)\\
&= (1,0,\dots,0)^\intercal\otimes(r_1s_1,\dots,r_1s_n)+\cdots+(0,\dots,0,1)^\intercal\otimes(r_ns_1,\dots,r_ns_n).
\end{aligned}
$$

We thus also see that $\tilde{g}$ is injective:

$$
\begin{aligned}
\tilde{g}((r_1,\dots,r_n)^\intercal\otimes(s_1,\dots,s_n)) = 0 &\implies \forall i,\forall j, r_is_j = 0\\
&\implies (1,0,\dots,0)^\intercal\otimes(r_1s_1,\dots,r_1s_n)\\
&\qquad +\cdots+(0,\dots,0,1)^\intercal\otimes(r_ns_1,\dots,r_ns_n) = 0\\
&\implies (r_1\dots,r_n)^\intercal\otimes(s_1,\dots,s_n) = 0.
\end{aligned}
$$

Since $\tilde{g}$ is $M_n(R)$-balanced, it is a bimodule homomorphism, and hence a bimodule isomorphism.

By Corollary 1.6.18, we see that R and $M_n(R)$ are Morita equivalent, as we saw earlier in Example 1.3.3.

In the following two lemmas and one proposition, we attempt to understand Definition 1.4.16 from a bicategorical perspective. That is, we explore what it means for a bimodule to be faithfully balanced in Bim.

Lemma 1.6.20. *The map $f\colon R \to \mathrm{End}_R(R_R)$ defined by $f(r) = \phi_r$, where ϕ_r is left multiplication by r, is a ring isomorphism.*

Proof. We first show that all the endomorphisms of R_R are left multiplications by elements of R. Let $\zeta \in \mathrm{End}_R(R_R)$. Then, for $s \in R$,

$$\zeta(s) = \zeta(1s) = \zeta(1)s = \phi_{\zeta(1)}(s).$$

Now we show that every left multiplication by an element of R is a right R-module homomorphism. Let $r, r' \in R$ and $m \in R_R$. Then

$$\phi_r(mr') = rmr' = \phi_r(m)r'.$$

Also, for $m' \in R_R$

$$\phi_r(m+m') = r(m+m') = rm + rm' = \phi_r(m) + \phi_r(m').$$

Finally, we show that f is a ring homomorphism. For $r, r' \in R$ and $m \in R_R$,

$$f(r+r')(m) = \phi_{r+r'}(m) = (r+r')m = rm + r'm = \phi_r(m) + \phi_{r'}(m) = f(r)(m) + f(r')(m);$$

$$f(rr')(m) = \phi_{rr'}(m) = rr'm = \phi_r(\phi_{r'}(m)) = (\phi_r \circ \phi_{r'})(m) = f(r)f(r')(m);$$

and

$$f(1)(m) = \phi_1(m) = 1m = m.$$

Since f is a ring homomorphism, every module endomorphism of R_R is a left multiplication by some element of R, and every left multiplication by some element of R is a module endomorphism, we see that f is a ring isomorphism. □

Lemma 1.6.21. *The map $f\colon R^{\mathrm{op}} \to \mathrm{End}_R({}_RR)$ defined by $f(r) = \psi_r$, where ψ_r is right multiplication by r, is a ring isomorphism.*

Proposition 1.6.22. *Given rings R and S, if ${}_RM_S$ is a faithfully balanced (R,S)-bimodule, then the maps $f\colon \mathrm{End}({}_{\mathbb{Z}}R_R) \to \mathrm{End}({}_{\mathbb{Z}}M_S)$ and $g\colon \mathrm{End}({}_{S^{\mathrm{op}}}S^{\mathrm{op}}_{\mathbb{Z}}) \to \mathrm{End}({}_RM_{\mathbb{Z}})$ defined for $r \in R$, $s \in S$, and $m \in M$ by*

$$f(\phi_r)(m) = r \cdot m \quad \text{and} \quad g(\psi_s)(m) = m \cdot s$$

are ring isomorphisms, where ϕ_r is left multiplication by r and ψ_s is right multiplication by s.

Proof. By the proofs of Lemma 1.6.20 and Lemma 1.6.21, it is clear that f and g are well-defined. Also, $\mathrm{End}({}_{\mathbb{Z}}R_R) = \mathrm{End}(R_R)$, $\mathrm{End}({}_{\mathbb{Z}}M_S) = \mathrm{End}(M_S)$, $\mathrm{End}({}_{S^{\mathrm{op}}}S^{\mathrm{op}}_{\mathbb{Z}}) = \mathrm{End}({}_{S^{\mathrm{op}}}S^{\mathrm{op}})$, and $\mathrm{End}({}_RM_{\mathbb{Z}}) = \mathrm{End}({}_RM)$. Note that f is the map from Lemma 1.6.20 composed with λ_M from Definition 1.4.16, both of which are ring isomorphisms, and g is the map from Lemma 1.6.21 composed with ρ_M from Definition 1.4.16, both of which are also ring isomorphisms. Thus f and g are both ring isomorphisms. □

We now quote two propositions from Rachel Brouwer who studied the use of Bim in Morita theory.

Proposition 1.6.23 ([Bro03, Proposition 2.4.1])**.** *Given rings R, and S, let ${}_RM_S$ be an (R,S)-bimodule. Then ${}_RM_S$ is an invertible morphism in Bim if and only if*

- *M_S is a progenerator for Mod_S; and*
- $R \cong \mathrm{End}_{S^{\mathrm{op}}}(M_S)$.

Proposition 1.6.24 ([Bro03, Subsection 2.4])**.** *Assuming one of Theorem 1.6.3 and Theorem 1.4.24, one can prove the other using Proposition 1.6.23.*

2 Dualities in Representation Theory

2.1 Group Representations: Connection to Modules

Here, we introduce group representations, and we show that they are just a certain type of module. We use this fact to study group representations using module theory.

Definition 2.1.1 (Group representation)**.** Given a group G and a field $\mathbb{K}$, a *(linear) representation* of G (over $\mathbb{K}$) is a group homomorphism

$$G \to \mathrm{GL}(V),$$

where V is a finite-dimensional vector space over $\mathbb{K}$ with $\dim(V) \geq 1$.

Definition 2.1.2 (Group algebra)**.** Given a group G and a field $\mathbb{K}$, the *group algebra* $\mathbb{K}[G]$ of G over $\mathbb{K}$ is the algebra with the elements of G as a basis and with multiplication on the basis elements given by group multiplication.

Note that $\mathbb{K}[G]$ has the multiplicative identity 1_G, meaning that $\mathbb{K}[G]$ can be seen as a ring. Note also that the elements of $\mathbb{K}[G]$ for any $\mathbb{K}$ and G are of the form $\sum_{g\in G} a_g g$. Multiplication on arbitrary elements of $\mathbb{K}[G]$ is given by the bilinearity of the multiplication on the basis elements:

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right)k.$$

Definition 2.1.3 (Endomorphism ring)**.** Given a vector space V, the *endomorphism ring* of V, denoted $\mathrm{End}(V)$, is the set of all linear endomorphisms of G where addition in $\mathrm{End}(V)$ is pointwise addition, and multiplication in $\mathrm{End}(V)$ is composition of linear maps.

Proposition 2.1.4 ([Web14, Proposition 1.2])**.** *Given a group G and a field $\mathbb{K}$, every $\mathbb{K}[G]$-module provides a representation of G over $\mathbb{K}$, and every representation of G over $\mathbb{K}$ induces a $\mathbb{K}[G]$-module action on V.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation on some $\mathbb{K}$-vector space V. Then

$$\left(\sum_{g\in G} a_g g\right) v = \sum_{g\in G} a_g \rho(g)(v)$$

defines a $\mathbb{K}[G]$-module action on V. We verify this claim. First, distributivity over the vector addition is satisfied:

$$\begin{aligned}
\left(\sum_{g\in G} a_g g\right)(v+v') &= \sum_{g\in G} a_g \rho(g)(v+v') \\
&= \sum_{g\in G} a_g(\rho(g)(v)+\rho(g)(v')) \quad \text{distributivity of matrix multiplication} \\
&= \sum_{g\in G} a_g\rho(g)(v) + a_g\rho(g)(v) \\
&= \sum_{g\in G} a_g\rho(g)(v) + \sum_{g\in G} a_g\rho(g)(v') \\
&= \left(\sum_{g\in G} a_g g\right)v + \left(\sum_{g\in G} a_g g\right)v'.
\end{aligned}$$

Distributivity over the group algebra addition is satisfied:

$$\begin{aligned}
\left(\sum_{g\in G} a_g g + \sum_{g\in G} b_g g\right)v &= \left(\sum_{g\in G} (a_g+b_g) g\right)v \\
&= \sum_{g\in G} (a_g+b_g)\rho(g)v \\
&= \sum_{g\in G} a_g\rho(g)v + b_g\rho(g)v \\
&= \sum_{g\in G} a_g\rho(g)v + \sum_{g\in G} b_g\rho(g)v \\
&= \left(\sum_{g\in G} a_g g\right)v + \left(\sum_{g\in g} b_g g\right)v.
\end{aligned}$$

The module action commutes with the group algebra multiplication:

$$\begin{aligned}
\left(\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right)\right) v &= \left(\sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right) k\right) v \\
&= \sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right)\rho(k)v \\
&= \sum_{gh\in G}\left(\sum_{g\in G} a_g\right)\left(\sum_{h\in H} b_h\right)\rho(g)\rho(h)v \\
&= \sum_{g\in G} a_g g\left(\sum_{h\in G} b_h\rho(h)v\right) \\
&= \sum_{g\in G} a_g g\left(\left(\sum_{h\in G} b_h h\right) v\right).
\end{aligned}$$

The identity preserves elements:

$$1_G v = \rho(1_G)v = v.$$

Thus, every representation induces a module action as specified above.

Now suppose that V is a $\mathbb{K}[G]$-module. The module action induces a ring homomorphism $\sigma\colon \mathbb{K}[G] \to \mathrm{End}(V)$ defined by

$$\sigma\left(\sum_{g\in G} a_g g\right)(v) = \left(\sum_{g\in G} a_g g\right) v.$$

Note that σ descends to a map $\tilde{\sigma}\colon G \to GL(V)$. First, for any $g \in G$, $\rho(g)$ has an inverse:

$$\rho(g)\rho(g^{-1}) = \rho(gg^{-1}) = \rho(1_G) = 1_{\mathrm{End}(V)}.$$

Thus, the range of $\tilde{\sigma}$ is at most $GL(V)$. Also, $\tilde{\sigma}$ is evidently a group homomorphism. Thus $\tilde{\sigma}$ is a representation on G. □

Definition 2.1.5 (Irreducible representation)**.** Given a group G and a field $\mathbb{K}$, the *irreducible representations* of G over $\mathbb{K}$ are those representations whose corresponding $\mathbb{K}[G]$-modules are simple.

Henceforth, given a group G and a field $\mathbb{K}$, the *representation* $\rho\colon G \to \mathrm{GL}(V)$ of G over $\mathbb{K}$ will refer to the corresponding $\mathbb{K}[G]$-module V given by the proof of Proposition 2.1.4.

Definition 2.1.6 (Direct sum of representations)**.** Given a group G, a field $\mathbb{K}$, and two representations V_1 and V_2, the *direct sum* $V_1 \oplus V_2$ is a representation of G whose action is given by

$$g(v_1, v_2) = (gv_1, gv_2).$$

Definition 2.1.7 (Tensor product of representations)**.** Given a group G, a field $\mathbb{K}$, and two representations V_1 and V_2, the *tensor product* $V_1 \otimes V_2$ is a representation whose action is given by

$$g(v_1 \otimes v_2) = (gv_1 \otimes gv_2).$$

Denote by $\mathfrak{S}_n$ the symmetric group on n symbols.

Theorem 2.1.8 (Maschke, [Lan02, Theorem 1.2])**.** *Let G be a finite group of order n, and let $\mathbb{K}$ be a field whose characteristic does not divide n. Then all $\mathbb{K}[G]$-modules are semisimple.*

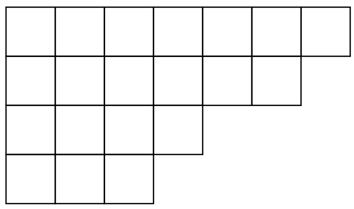
Corollary 2.1.9. *The complex representations of $\mathfrak{S}_n$ are decomposable into direct sums of irreducible representations.*

Definition 2.1.10. A *Young diagram* is a collection of boxes in left-justified rows with the number of boxes in each row less than or equal to the number of boxes in the previous row.

Given a positive integer n, a partition of n is notated here as a weakly decreasing, eventually zero sequence $(\lambda_1, \lambda_2, \ldots, 0, \ldots)$ of positive integers such that $\sum_{i=1}^{\infty} \lambda_i = n$.

Given a Young diagram with n boxes and k rows, and letting κ_i be the number of boxes in the i-th row, the sequence $(\kappa_1, \ldots, \kappa_k, 0, \ldots)$ is a partition of n. Conversely, a partition $(\lambda_1, \ldots, \lambda_k, 0, \ldots)$ of n gives a Young diagram whose i-th row has λ_i boxes.

Example 2.1.11. The partition $(7, 6, 4, 3, 0, \ldots)$ corresponds to the following Young diagram.



Proposition 2.1.12 ([Ful97, Proposition 7.2.1])**.** *The Young diagrams with n boxes are in bijection with the irreducible complex representations of $\mathfrak{S}_n$.*

Definition 2.1.13 (Polynomial representation)**.** A representation V of $\mathrm{GL}(V)$ over $\mathbb{C}$ is called a *polynomial representation* if, for all $g \in \mathrm{GL}(V)$, the entries of the matrix given by the action of g on V are polynomial in the matrix entries of g.

Proposition 2.1.14 ([Ful97, Theorem 8.2.2])**.** *The Young diagrams with at most n rows are in bijection with the irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$.*

It is a well-known fact that the general linear group $\mathrm{GL}_n(\mathbb{C})$ is an example of what is called a reductive group. A key characteristic of reductive groups is that their representations over fields of characteristic 0 are decomposable into direct sums of irreducible representations. See [Bor91, p. 273] for some discussion on the latter statement.

Lemma 2.1.15 (Schur, [Coh03, Lemma 6.3.2]). *Let $\mathbb{K}$ be an algebraically closed field, A a $\mathbb{K}$-algebra, and U, V any two simple A-modules, finite-dimensional over $\mathbb{K}$. Then*

$$\operatorname{Hom}_A(U,V) \cong \begin{cases} \mathbb{K} & \text{if } U \cong V \\ \{0\} & \text{otherwise.} \end{cases}$$

where the first isomorphism is of algebras if $U = V$.

The following proposition is key in the proof of Proposition 2.2.2.

Proposition 2.1.16. *Let G and H be two groups such that ${}_{\mathbb{C}[G]}\mathsf{Mod}$ and ${}_{\mathbb{C}[H]}\mathsf{Mod}$ are semisimple module categories with the same number of isomorphism classes of simple objects. Then $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are Morita equivalent.*

Proof. Since categories are equivalent to their skeletons by Proposition 1.1.15, it suffices to show that a skeleton $\mathcal{C}$ of ${}_{\mathbb{C}[G]}\mathsf{Mod}$ is equivalent to a skeleton $\mathcal{D}$ of ${}_{\mathbb{C}[H]}\mathsf{Mod}$.

Since ${}_{\mathbb{C}[G]}\mathsf{Mod}$ and ${}_{\mathbb{C}[H]}\mathsf{Mod}$ are semisimple module categories, every object in either category is decomposable into a direct sum of simple objects. Let $V_1, V_2, \dots, V_n$ be the simple objects of $\mathcal{C}$, and let $W_1, W_2, \dots, W_n$ be the simple objects of $\mathcal{D}$. Let $F\colon {}_{\mathcal{C}}\mathsf{Mod} \to {}_{\mathcal{D}}\mathsf{Mod}$ be a functor whose map on objects is defined by

$$F\left(\bigoplus_{i=1}^{n} V_i^{\oplus n_i}\right) = \bigoplus_{i+1}^{n} W_i^{\oplus n_i}$$

for $n_i \in \mathbb{N}$. It is seen immediately that F is surjective.

Let

$$\iota_{a,b}\colon V_a \to \bigoplus_{i=1}^{n} V_i^{\oplus n_i}$$

for $1 \le a \le n$ and $1 \le b \le n_i$ be the canonical injection and let

$$\pi_{a,b}\colon \bigoplus_{i=1} V_i^{\oplus m_j} \to V_a$$

for $1 \le a \le n$ and $1 \le b \le m_j$ be the canonical projection. Then for every map

$$f \in \operatorname{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^{n} V_i^{\oplus n_i}, \bigoplus_{j=1}^{n} V_j^{\oplus m_j}\right)$$

and for every $1 \le i \le n, 1 \le j \le n, 1 \le a \le n_i$, and $1 \le b \le m_j$, we get a collection of maps

$$\eta_{i,j,a,b}(f) = \pi_{j,b} \circ f \circ \iota_{i,a} \in \operatorname{Hom}_{\mathbb{C}[G]}(V_i, V_j).$$

It is noteworthy that η is compatible with composition: for

$$g \in \operatorname{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{j=1}^{n} V_j^{\oplus m_j}, \bigoplus_{k=1}^{n} V_k^{\oplus \ell_k}\right)$$

and $1 \le c \le \ell_k$,

$$\eta_{i,k,a,c}(g \circ f) = \pi_{k,c} \circ g \circ f \circ \iota_{i,a} = \sum_{j,b} \pi_{k,c} \circ g \circ \iota_{j,b} \circ \pi_{j,b} \circ f \circ \iota_{i,a} = \sum_{j,b} \eta_{j,k,b,c}(g) \circ \eta_{i,j,a,b}(f).$$

Further, we know by Lemma 1.2.25 and Lemma 1.2.26 that

$$f \mapsto \bigoplus_{i,j=1}^{n} \bigoplus_{a}^{n_i} \bigoplus_{b}^{m_j} \eta_{i,j,a,b}(f)$$

is an abelian group isomorphism.

By Lemma 2.1.15, we know that $\eta_{i,j,a,b}(f) = 0$ when $i \neq j$ and that there is an isomorphism of algebras $\phi_i\colon \mathrm{End}_{\mathbb{C}[G]}(V_i) \to \mathrm{End}_{\mathbb{C}[H]}(W_i)$ for $1 \le i \le n$. By again applying Lemma 1.2.25 and Lemma 1.2.26, we find an isomorphism

$$\mathrm{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^{n} V_i^{\oplus n_i}, \bigoplus_{j=1}^{n} V_j^{\oplus m_j}\right) \to \mathrm{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^{n} W_i^{\oplus n_i}, \bigoplus_{j=1}^{n} W_j^{\oplus m_j}\right),$$

and we set the map on morphisms of F to be this map:

$$F(f) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \phi_i(\eta_{i,i,j,k}(f))$$

This map is compatible with composition as well:

$$\begin{aligned}
F(g \circ f) &= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \phi_i(\eta_{i,i,j,k}(g \circ f)) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \phi_i(\eta_{i,i,j,k}(g)) \circ \phi_i(\eta_{i,i,j,k}(f)) \\
&= \left(\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \phi_i(\eta_{i,i,j,k}(g))\right) \circ \left(\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \phi_i(\eta_{i,i,j,k}(f))\right) \\
&= F(g) \circ F(f).
\end{aligned}$$

Note that the above composition of sums makes sense because these sums are coordinate-wise, and the composition is strictly across these coordinates; in other words, the composition is zero when the coordinates do not match.

Since the map of F on morphisms is an isomorphism, we see that F is fully faithful, as well as surjective as mentioned above. Thus, by Theorem 1.1.18, F is an equivalence of categories. □

2.2 Schur-Weyl Duality

In this section, we treat the classical Schur-Weyl duality theorem (Theorem 2.2.1) and we show that it exhibits a Morita equivalence.

The notation now defined will be carried onto the statement of the next theorem. Let V denote a complex vector space of dimension n. Let $V^{\otimes r}$ denote the r-fold tensor product of V. We endow $V^{\otimes r}$ with the following $\mathbb{C}[\mathrm{GL}(V)]$-action: for $g \in \mathrm{GL}(V)$,

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = g(v_1) \otimes g(v_2) \otimes \cdots \otimes g(v_r).$$

We also give $V^{\otimes r}$ the following $\mathbb{C}[\mathfrak{S}_r]$-action: for $\sigma \in \mathfrak{S}_r$,

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

It is evident that the $\mathbb{C}[\mathrm{GL}(V)]$-action and $\mathbb{C}[\mathfrak{S}_r]$-action commute with one another, making $V^{\otimes r}$ a $(\mathbb{C}[\mathrm{GL}(V)], \mathbb{C}[\mathfrak{S}_r])$-bimodule. For a given Young diagram D, let U^D be the irreducible polynomial representation of $\mathrm{GL}(V)$ corresponding to the Young diagram D, and let W^D be the irreducible complex representation of $\mathfrak{S}_r$ on $V^{\otimes r}$ corresponding to the Young diagram D.

Theorem 2.2.1 (Schur-Weyl duality, [How95, Section 2.4.3])**.** *Under the joint action of* $\mathrm{GL}(V)$ *and* $\mathfrak{S}_r$*, the space* $V^{\otimes r}$ *decomposes into a direct sum*

$$V^{\otimes r} \cong \bigoplus_D U^D \otimes W^D,$$

where D runs over Young diagrams with r boxes and at most n rows.

Proposition 2.2.2. *Given the algebra homomorphisms*

$$\mathbb{C}[\mathrm{GL}(V)] \xrightarrow{\alpha} \mathrm{End}(V^{\otimes r}) \xleftarrow{\beta} \mathbb{C}[\mathfrak{S}_r]$$

induced by the module actions of $\mathbb{C}[\mathrm{GL}(V)]$ *and* $\mathbb{C}[\mathfrak{S}_r]$ *on* $V^{\otimes r}$*, we have that* ${}_{\mathrm{im}\,\alpha}\mathsf{Mod}$ *is Morita equivalent to* ${}_{\mathrm{im}\,\beta}\mathsf{Mod}$*.*

Proof. Let $f \in \bigoplus_D \mathrm{End}(U_D)$. Then $f = \bigoplus_D f_D$ for maps $f_D \in \mathrm{End}(U_D)$, and we can map

$$\omega\colon f \mapsto \bigoplus_D (f_D \otimes \mathrm{id}_{W_D}).$$

The map ω is injective: suppose $g \in \bigoplus_D \mathrm{End}(U_D)$, $g = \bigoplus_D g_D$ and that $f_D \otimes \mathrm{id}_{W_D} = g_D \otimes \mathrm{id}_{W_D}$ for all D. Then for $x \in U_D$, $y \in W_D$ nonzero,

$$\begin{aligned} & f_D \otimes \mathrm{id}_{W_D} = g_D \otimes \mathrm{id}_{W_D} \\ \implies & f_D(x) \otimes y = g_D(x) \otimes y \\ \implies & (f_D(x) - g_D(x)) \otimes y = 0. \end{aligned}$$

However, if $f_D(x) - g_D(x)$ is nonzero, then we can find a basis of U_D such that $f_D(x) - g_D(x)$ is a basis vector. Similarly, y can be considered a basis vector of W_D. However, this would allow $(f_D(x) - g_D(x)) \otimes y$ to be a basis vector for $U_D \otimes W_D$, and basis vectors cannot be 0. This is a contradiction. Thus $f_D(x) - g_D(x) = 0$ for all D, which implies $f = g$.

We see that

$$\Omega = \{\omega(f) \colon f \in \bigoplus_D \operatorname{End}(U_D)\}$$

must consist entirely of maps of $\bigoplus_D \operatorname{End}(U_D \otimes W_D)$ that commute with the $\mathbb{C}[\mathfrak{S}_r]$-action on $\bigoplus U_D \otimes W_D$. That is, Ω commutes with $\operatorname{im}\beta$.

As left $\mathbb{C}[\mathrm{GL}(V)]$-modules,

$$U_D \otimes W_D \cong (U_D \otimes \mathbb{C}w_1) \oplus (U_D \otimes \mathbb{C}w_2) \oplus \cdots \oplus (U_D \otimes \mathbb{C}w_{\dim W_D}) \cong U_D^{\oplus \dim W_D},$$

where $\{w_1, w_2, \ldots, w_{\dim W_D}\}$ is a basis for W_D. Thus,

$$\bigoplus_D U_D \otimes W_D \cong \bigoplus_D U_D^{\oplus \dim W_D}.$$

Therefore, we have the isomorphism

$$\operatorname{End}\left(\bigoplus_D U_D \otimes W_D\right) \cong \operatorname{End}\left(\bigoplus_D U_D^{\oplus \dim W_D}\right).$$

Pulling out the direct sums with Lemma 1.2.25 and Lemma 1.2.26, we obtain

$$\operatorname{End}\left(\bigoplus_D U_D^{\oplus \dim W_D}\right) \cong \bigoplus_{D,D'} \operatorname{Hom}\left(U_D, U_{D'}\right)^{\oplus (\dim W_D)(\dim W_{D'})}.$$

We know that the U_D are simple modules, however, and by Lemma 2.1.15, we see that $\operatorname{Hom}(U_D, U_{D'}) \cong 0$ when $D \neq D'$. Thus,

$$\begin{aligned}\bigoplus_{D,D'} \operatorname{Hom}\left(U_D, U_{D'}\right)^{\oplus (\dim W_D)(\dim W_{D'})} &\cong \bigoplus_D \operatorname{Hom}\left(U_D, U_D\right)^{\oplus (\dim W_D)(\dim W_D)} \\ &\cong \bigoplus_D \operatorname{Hom}\left(U_D^{\oplus \dim W_D}, U_{D'}^{\oplus \dim W_D}\right) \\ &\cong \bigoplus_D \operatorname{End}\left(U_D^{\oplus \dim W_D}\right).\end{aligned}$$

We wish to show that every map in $\operatorname{im}\alpha$ is of the form $\bigoplus_D (f_D \otimes \mathrm{id}_{W_D})$. Let $(F_D)_D$ be a tuple of endomorphisms $F_D \in \operatorname{End}\left(U_D^{\oplus \dim W_D}\right) \cong \operatorname{End}(U_D \otimes W_D)$. Then $F_D(u \otimes w) = \sum_{i,j} a_i u_i \otimes w_j$ for any $u \in U$, $w \in W$.

Note that every W_D is a right $\mathbb{C}[\mathfrak{S}_r]$-module. Theorem 1.2.30 tells us that for any of the W_D, and for any $\gamma \in \operatorname{End}(W_D)$, and for any basis X, we can pick $s \in \mathbb{C}[\mathfrak{S}_r]$ such that

$ws = \gamma(w)$ for $w \in X$. Without loss of generality, let $w = w_1$. Let γ be the projection onto $\mathbb{C}w_1 = \mathbb{C}w$, and let s be the element of $\mathbb{C}[\mathfrak{S}_r]$ which corresponds to γ. Then

$$F_D \circ (\mathrm{id}_{U_D} \otimes s)(u \otimes w) = F_D(u \otimes w_1) = F_D(u \otimes w) = \sum_i a_i u_i \otimes w_i,$$

and

$$(\mathrm{id}_{U_D} \otimes s) \circ F_D(u \otimes w) = (\mathrm{id}_{U_D} \otimes s)(\sum_i a_i u_i \otimes w_i) = a_1 u_1 \otimes w_1 = a_1 u_1 \otimes w.$$

Since F_D must commute with the action of $\mathbb{C}[\mathfrak{S}_r]$ on W_D, for any $w \in W_D$, it happens that $F_D(u \otimes w) = a_1 u_1 \otimes w$, and we obtain that, as an endomorphism of $U_D \otimes W_D$, we will have

$$F_D(u \otimes w) = u' \otimes w$$

for any $w \in W_D$, and for any $u \in U_D$. Hence $F_D = f_D \otimes \mathrm{id}_{W_D}$ for some $f_D \in \mathrm{End}(U)$. Thus ω surjects onto $\mathrm{End}\left(U_D^{\oplus \dim W_D}\right)$. This makes

$$\omega \colon \bigoplus_D \mathrm{End}(U_D) \to \bigoplus_D \mathrm{End}\left(U_D^{\oplus \dim W_D}\right)$$

an isomorphism of algebras.

Now, since $\mathrm{im}\,\alpha \cong \bigoplus_D \mathrm{End}\left(U_D^{\oplus \dim W_D}\right)$, we see that by the isomorphism ω,

$$\mathrm{im}\,\alpha \cong \bigoplus_D \mathrm{End}(U_D),$$

and by similar reasoning,

$$\mathrm{im}\,\beta \cong \bigoplus_D \mathrm{End}(W_D).$$

We know that $\mathrm{End}(U_D)$ for each D is a simple ring as $\mathrm{End}(U_D) \cong M_{\dim U_D}(\mathbb{C})$, $M_{\dim U_D}(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ by Example 1.3.3, $\mathbb{C}$ is a simple ring as $\mathbb{C}$ is a field, and by Proposition 1.2.23, $\mathbb{C}$ has one simple module. The simple modules of $\bigoplus_D \mathrm{End}(U_D)$ are in correspondence with the maximal ideals of $\bigoplus_D \mathrm{End}(U_D)$ by Proposition 1.2.23 which are in turn enumerated by the Young diagrams D as the maximal ideals are of the form

$$\mathrm{End}(U_{D_1}) \oplus \cdots \oplus \mathrm{End}(U_{D_{i-1}}) \oplus \{0\} \oplus \mathrm{End}(U_{D_{i+1}}) \oplus \cdots \oplus \mathrm{End}(U_{D_d})$$

where d is the number of Young diagrams. The same reasoning applies to show that the simple modules of $\bigoplus_D \mathrm{End}(W_D)$ are enumerated by the Young diagrams D. Thus ${}_{\mathrm{im}\,\alpha}\mathsf{Mod}$ and ${}_{\mathrm{im}\,\beta}\mathsf{Mod}$ are semisimple module categories with the same number of isomorphism classes of simple modules. Finally, by Proposition 2.1.16, we see that since ${}_{\mathrm{im}\,\alpha}\mathsf{Mod}$ and ${}_{\mathrm{im}\,\beta}\mathsf{Mod}$ have the same number of isomorphism classes of simple modules, $\mathrm{im}\,\alpha$ and $\mathrm{im}\,\beta$ must be Morita equivalent. $\square$

It is of note that $\mathrm{im}\,\beta$ in the above proposition is called the *Schur algebra* $S_{\mathbb{C}}(n,r)$, and has been the subject of extensive study.

2.3 $GL_n - GL_m$ and Skew $GL_n - GL_m$ Duality

We now discuss two other dualities found in [How95] in which one finds Morita equivalence.

Definition 2.3.1 (Tensor algebra)**.** Given a vector space V, we call the associative algebra $\mathfrak{T}(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ with associative product

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m$$

the *tensor algebra* on V.

Definition 2.3.2 (Symmetric algebra)**.** Given a vector space V, let I be the two-sided ideal of $\mathfrak{T}(V)$ generated by $x \otimes y - y \otimes x$ for all $x, y \in V$. Then we call $\mathfrak{S}(V) = \mathfrak{T}(V)/I$ the *symmetric algebra* on V.

Definition 2.3.3 (Exterior algebra)**.** Given a vector space V, let I be the two-sided ideal of $\mathfrak{T}(V)$ generated by $x \otimes y + y \otimes x$ for all $x, y \in V$. Then we call $\Lambda(V) = \mathfrak{T}(V)/I$ the *exterior algebra* on V.

Note that given a group algebra $\mathbb{K}[G]$ and a (left) $\mathbb{K}[G]$-module V, the tensor algebra $\mathfrak{T}(V)$ has a natural left $\mathbb{K}[G]$-action:

$$g(v_1 \otimes \cdots \otimes v_n) = (gv_1) \otimes \cdots \otimes (gv_n)$$

for all $g \in G$, which is extended by linearity. Further, the ideal $(x \otimes y - y \otimes x) \subset \mathfrak{T}(V)$ is closed under the R-action on the $\mathfrak{T}(V)$.

$$r(x \otimes y - y \otimes x) = r(x \otimes y) - r(y \otimes x) = (rx) \otimes (ry) - (ry) \otimes (rx).$$

Thus, the R-action on $\mathfrak{T}(V)$ descends to an R-action on the symmetric algebra $\mathfrak{S}(V)$. In a similar fashion, the R-action on $\mathfrak{T}(V)$ descends to an R-action on the exterior algebra $\Lambda(V)$.

The notation developed here carries on into the next two theorems. Let U and V be two complex vector spaces. We give $U \otimes V$ the following $\mathbb{C}[\mathrm{GL}(U)]$-action: for $g \in \mathrm{GL}(U)$,

$$g(u \otimes v) = g(u) \otimes v.$$

We also give $U \otimes V$ a right $\mathbb{C}[\mathrm{GL}(V)]$-action: for $h \in \mathrm{GL}(V)$,

$$(u \otimes v)h = u \otimes h(v).$$

The $\mathbb{C}[\mathrm{GL}(U)]$- and $\mathbb{C}[\mathrm{GL}(V)]$-actions commute, making $U \otimes V$ a $(\mathbb{C}[\mathrm{GL}(U)], \mathbb{C}[\mathrm{GL}(V)])$-module. We can thus treat $\mathfrak{S}(U \otimes V)$ and $\Lambda(U \otimes V)$ as $(\mathbb{C}[\mathrm{GL}(U)], \mathbb{C}[\mathrm{GL}(V)])$-modules. Let U^D and V^D be the irreducible polynomial representations of $\mathrm{GL}(U)$ and $\mathrm{GL}(V)$, respectively, corresponding to the Young diagrams D.

Theorem 2.3.4 ((GL_n,GL_m)-duality, [How95, 15-16])**.** *As a* $(\mathbb{C}[\mathrm{GL}(U)],\mathbb{C}[\mathrm{GL}(V)])$*-module,* $\mathfrak{S}(U\otimes V)$ *decomposes into a direct sum*

$$\mathfrak{S}(U\otimes V)\cong\bigoplus_D U^D\otimes V^D,$$

where D runs over all Young diagrams with at most $\min\{\dim U,\dim V\}$ *rows.*

Corollary 2.3.5. *Given the algebra homomorphisms*

$$\mathbb{C}[\mathrm{GL}(U)]\xrightarrow{\alpha}\mathrm{End}(\mathfrak{S}(U\otimes V))\xleftarrow{\beta}\mathbb{C}[\mathrm{GL}(V)]$$

induced by the module actions of $\mathbb{C}[\mathrm{GL}(V)]$ *and* $\mathbb{C}[\mathrm{GL}(V)]$ *on* $\mathfrak{S}(U\otimes V)$*, we have that* ${}_{\mathrm{im}\,\alpha}\mathsf{Mod}$ *is equivalent to* ${}_{\mathrm{im}\,\beta}\mathsf{Mod}$*.*

Proof. This is justified in the same way as Proposition 2.2.2. □

Given a Young diagram D, its transpose $D^\intercal$ is the Young diagram whose columns are the rows of D (and thus whose rows are the columns of D).

Theorem 2.3.6 (Skew (GL_n,GL_m)-duality, [How95, 50])**.** *As a* $(\mathbb{C}[\mathrm{GL}(U)],\mathbb{C}[\mathrm{GL}(V)])$*-module,* $\Lambda(U\otimes V)$ *decomposes into a direct sum*

$$\Lambda(U\otimes V)\cong\bigoplus_D U^D\otimes V^{D^\intercal},$$

where D runs over all Young diagrams with at most n rows and with rows of length at most m.

Corollary 2.3.7. *Given the algebra homomorphisms*

$$\mathbb{C}[\mathrm{GL}(U)]\xrightarrow{\alpha}\mathrm{End}(\Lambda(U\otimes V))\xleftarrow{\beta}\mathbb{C}[\mathrm{GL}(V)]$$

induced by the module actions of $\mathbb{C}[\mathrm{GL}(V)]$ *and* $\mathbb{C}[\mathrm{GL}(V)]$ *on* $\Lambda(U\otimes V)$*, we have that* ${}_{\mathrm{im}\,\alpha}\mathsf{Mod}$ *is equivalent to* ${}_{\mathrm{im}\,\beta}\mathsf{Mod}$*.*

Proof. This is justified in the same way as Proposition 2.2.2. □

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