QUIVERS AND THREE-DIMENSIONAL LIE ALGEBRAS

STÉPHANE GUÉRIN AND JEFFREY PIKE

Abstract. In this paper, we review the classification of Lie algebras of dimension at most three, and focus on a specific family of such Lie algebras. We describe their (modified) universal enveloping algebras in terms of the path algebras of certain quivers, denoted by $Q_{m,n}$ and $Q_{\infty\times\infty}$. We then study the representation theory of the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$, and describe functors which embed the categories of representations of $Q_{m,n}$ and $Q_{\infty\times\infty}$ inside categories of representations of double quivers of type A_∞ and $A_n^{(1)}$, which are well understood. If we consider only finite dimensional representations that satisfy certain relations, these functors send the representations of $Q_{m,n}$ or $Q_{\infty\times\infty}$ to points of specific varieties known as Lusztig's quiver varieties.

Introduction

The study of Lie groups and their corresponding Lie algebras has been an important part of modern mathematics since the theory was first developed in the 19^{th} century by M. Sophus Lie. Indeed, Lie algebras play a fundamental role across diverse areas of mathematics, such as geometry, topology, and of course group theory. A natural way to study Lie algebras is by their representation theory; that is, the ways in which one can 'represent' a Lie algebra by operators such that the Lie bracket is the usual commutator. In the late 20^{th} century, the mathematician Pierre Gabriel discovered a beautiful relationship between the root systems of Lie algebras [4], which play a vital role in their representation theory, and the representations of *quivers*, which are directed graphs. This result fuelled further research of the connection between Lie theory and quivers, as the correspondence between the two allows for a more intuitive and geometric means of studying Lie algebras. In the present work, we study the representation theory of a particular collection of Lie algebras by first relating them to certain quivers.

The *Euclidean group* is the group of isometries of $\mathbb{R}^2$ having determinant 1, and the *Euclidean algebra* is the complexification of its Lie algebra. The Euclidean group is one of the oldest and most studied examples of a group: it was studied implicitly even before the notion of a group was formalized, and it has applications not only throughout mathematics but in quantum mechanics, relativity, and other areas of physics as well. In [8], it is shown that the category of representations of the Euclidean algebra is equivalent to the category of representations of the preprojective algebras of quivers of type A_∞. In the current paper, we consider a family of 3-dimensional Lie algebras depending on a continuous parameter $\mu \in \mathbb{C}^*$; we denote the members of this family by L_μ. We separate the discussion of the family L_μ into two cases: $\mu \in \mathbb{Q}^*$, and $\mu \in \mathbb{C} \setminus \mathbb{Q}$. When $\mu \in \mathbb{Q}^*$, we show that the category of representations of L_μ is equivalent to the category of representations of a quiver that we denote $Q_{m,n}$, where $\mu = \frac{n}{m}$. We then show that the category of representations of $Q_{m,n}$ can be embedded inside the category of representations of the preprojective algebra of the affine quiver of type $A_{m+n}^{(1)}$. It can be shown that the Euclidean algebra is isomorphic to the Lie algebra L_{-1}, and so if $\mu = -1$ then $m = -n$ so that this agrees with what is presented in [8]. Thus the current work can be thought of as a generalization of some of the results of that paper. In the case $\mu \in \mathbb{C} \setminus \mathbb{Q}$, we present an equivalence of categories between the representations of L_μ and the

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representations of a quiver denoted $Q_{\infty\times\infty}$. Then, analogous to the rational case, we show how the representations of $Q_{\infty\times\infty}$ form a subcategory of the category of representations of the preprojective algebra of the quiver of type A_∞.

In Section 1 we introduce some elementary notions from Lie theory and review a classification of all Lie algebras having dimension at most 3, and we then introduce the family L_μ of 3-dimensional Lie algebras that will be studied in the rest of the paper. In Section 2 we describe the (modified) universal enveloping algebras $\widetilde{U}_{m,n}$ and $\widetilde{U}$ associated with the Lie algebras L_μ. We then define the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$ in Section 3, and establish isomorphisms between the path algebras of these quivers and the modified enveloping algebras of the Lie algebras L_μ. Finally, in Section 4, we define the functors $\mathcal{G} : \mathrm{rep}(Q_{m,n}) \to \mathrm{rep}(\widehat{Q}_{m+n})$ and $\mathcal{F} : \mathrm{rep}(Q_{\infty\times\infty}) \to \mathrm{rep}(Q_\infty)$, and we show that both $\mathcal{G}$ and $\mathcal{F}$ are additive, faithful, and exact but neither is full or essentially surjective.

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1. Lie Algebras and the Family L_μ

1.1. **Lie Algebras.** A *Lie algebra* over a field F is an F-vector space, L, together with a bilinear map, the *Lie bracket*,

$$\begin{aligned} L \times L &\to L \\ (x, y) &\mapsto [x, y] \end{aligned}$$

satisfying the following properties:

$$\begin{aligned} [x, x] &= 0 \qquad \forall x \in L, \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \qquad \forall x, y, z \in L \qquad \text{(the Jacobi identity).} \end{aligned}$$

A Lie algebra is said to be *abelian* if $L = Z(L)$, where $Z(L) = \{x \in L \mid [x, y] = 0 \quad \forall y \in L\}$ is the centre of L. We define the *derived algebra* of a Lie algebra L as $L' = \mathrm{Span}\{[x, y] \mid x, y \in L\}$.

Given two Lie algebras, L_1 and L_2, a map $\varphi : L_1 \to L_2$ is a *Lie algebra homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for every $x, y \in L_1$. An important Lie algebra homomorphism is the *adjoint homomorphism*, $\mathrm{ad} : L \to \mathrm{End}(L)$ defined by $(\mathrm{ad}\, x)(y) := [x, y]$.

By a *representation* of a Lie algebra L, we shall mean a homomorphism $\varphi : L \to \mathfrak{gl}(V)$ for some vector space V. Here $\mathfrak{gl}(V)$ denotes the Lie algebra of endomorphisms of V with Lie bracket given by $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$. If $\varphi : L \to \mathfrak{gl}(V)$ is a representation of a Lie algebra L, then V is an *L-module*. By the usual abuse of terminology, we will often use the terms representation and module interchangeably.

1.2. **Lie Algebras of Low Dimension.** In this section we study complex Lie algebras of dimension at most 3. We review the classification of such Lie algebras, and then study one particular family of 3-dimensional Lie algebras. This family depends on a continuous parameter $\mu \in \mathbb{C}$, and we denote its members by L_μ. The Lie algebras L_μ are not as well understood as the other 3-dimensional complex Lie algebras, and their representation theory is the subject of the current paper.

First, we note that every 1-dimensional Lie algebra is abelian. It is not difficult to see that two abelian Lie algebras are isomorphic if and only if they have the same dimension, and hence all 1-dimensional Lie algebras are isomorphic. In the 2-dimensional case, there is a unique (up to isomorphism) non-abelian Lie algebra, which has a basis $\{x, y\}$ such that its Lie bracket is described by $[x, y] = x$. See for example [2, Theorem 3.1] for a proof of this statement.

In dimension 3, all cases of non-abelian Lie algebras can be classified by relating L' to $Z(L)$. First we will consider the cases $\dim L' = 1$ and $\dim L' = 3$.

Lemma 1.1 ([2, Section 3.2])**.** *There are unique 3-dimensional Lie algebras having the following properties:*

(i) $\dim L' = 1$ *and* $L' \subseteq Z(L)$. *This Lie algebra is known as the* Heisenberg algebra*.*
(ii) $\dim L' = 1$ *but* $L' \nsubseteq Z(L)$. *This Lie algebra is the direct sum of the 2-dimensional non-abelian Lie algebra with the 1-dimensional Lie algebra.*
(iii) $L' = L$. *In this case,* $L = \mathfrak{sl}(2, \mathbb{C})$, *the Lie subalgebra of* $\mathfrak{gl}(2, \mathbb{C}) := \mathfrak{gl}(\mathbb{C}^2)$ *consisting of trace zero operators.*

We will now consider the case where $\dim L = 3$ and $\dim L' = 2$. Let $\{y, z\}$ be a basis of L' and let $x \in L \setminus L'$. We now have a basis of L, $\{x, y, z\}$. We need the following lemma.

Lemma 1.2 ([2, Lemma 3.3])**.** *Let L be a Lie algebra such that* $\dim L = 3$ *and* $\dim L' = 2$, *and let* $x \in L \setminus L'$. *Then:*

(i) L' *is abelian.*
(ii) The map $\operatorname{ad} x : L' \to L'$*is an isomorphism.*

We can separate Lie algebras L having the properties $\dim L = 3$ and $\dim L' = 2$ into two cases. The first happens when there is some $\beta \in L \setminus L'$ such that $\operatorname{ad} \beta : L' \to L'$ is diagonalisable. The second case occurs if $\operatorname{ad} x : L' \to L'$ is not diagonalisable for any $x \in L \setminus L'$.

In the latter case, we get that the Jordan canonical form of the matrix of $\operatorname{ad} x$ must be a single 2×2 Jordan block. Since $\operatorname{ad} x$ is an isomorphism it must have an eigenvector with nonzero eigenvalue, and by proper scaling of x we may assume that this eigenvalue is 1. Thus the Jordan form of the matrix of $\operatorname{ad} x$ acting on L' is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which completely determines the Lie algebra L.

The case of interest for us will be the one where there is some $\beta \notin L'$ such that the map $\operatorname{ad} \beta : L' \to L'$ is diagonalisable. We will choose basis $\{\alpha_1, \alpha_2\}$ of L' consisting of eigenvectors of $\operatorname{ad} \beta$. Then the associated eigenvalues of α_1 and α_2 must be nonzero by part (ii) of Lemma 1.2.

Since α_1 and α_2 are eigenvectors of $\operatorname{ad} \beta$, $[\beta, \alpha_1] = \eta \alpha_1$ and $[\beta, \alpha_2] = \mu \alpha_2$ for some $\eta, \mu \in \mathbb{C}^*$. We have $[\eta^{-1} \beta, \alpha_1] = \alpha_1$, therefore, with proper scaling, we may assume that $\eta = 1$. With respect to the basis $\{\alpha_1, \alpha_2\}$, $\operatorname{ad} \beta : L' \to L'$ has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}. \tag{1.2.1}$$

We will call this Lie algebra L_μ. Furthermore, by [2, Exercise 3.2] we can see that $L_\mu \cong L_\nu \Leftrightarrow \mu = \nu$ or $\mu = \nu^{-1}$. This is true for all $\mu \in \mathbb{C}^*$. First we will focus on the case where $\mu \in \mathbb{Q}^*$. The matrix of $\operatorname{ad} x$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{m} \end{pmatrix},$$

where $\mu = \frac{n}{m}$. With a simple change of basis we get

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}. \tag{1.2.2}$$

Therefore, L_μ has basis $\{\alpha_1, \beta, \alpha_2\}$ and commutation relations

$$[\beta, \alpha_1] = m\alpha_1, \qquad [\beta, \alpha_2] = n\alpha_2, \qquad [\alpha_1, \alpha_2] = 0. \tag{1.2.3}$$

Remark 1.3. *A special case of this algebra occurs when $\mu = -1$ and we get L_{-1} which is the* Euclidean algebra*. See* [8, Section 2] *for a discussion of this Lie algebra in the same context as the current paper.*

Remark 1.4. *Since $\mu^{-1} = (\frac{n}{m})^{-1} = \frac{n}{m}$, we have $L_{\frac{n}{m}} \cong L_{\frac{n}{m}}$.*

On the other hand, if $\mu \in \mathbb{C} \setminus \mathbb{Q}$ then by (1.2.1) the commutation relations are

$$[\beta, \alpha_1] = \alpha_1, \qquad [\beta, \alpha_2] = \mu\alpha_2, \qquad [\alpha_1, \alpha_2] = 0. \tag{1.2.4}$$

2. The Universal Enveloping Algebra of L_μ and its Representations

2.1. **Universal Enveloping Algebras.** If L is a Lie algebra, then the *universal enveloping algebra* of L is the pair (U, i), where U is a unital associative algebra and $i : L \to U$ is a map satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x) \quad \forall x, y \in L. \tag{2.1.1}$$

Moreover, the pair (U, i) is *universal* with respect to this property. That is, for any pair (U', i') satisfying (2.1.1) there exists a unique homomorphism $\varphi : U \to U'$ such that $\varphi i = i'$. Since we have defined the universal enveloping algebra by a universal property, it is unique up to isomorphism. Furthermore, this definition makes it clear that the category of representations of a Lie algebra L is equivalent to the category of representations of its universal enveloping algebra U. If we denote by T the tensor algebra of the Lie algebra L and by I_T the two sided ideal of T generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in L$, then it can be shown that $U \cong T/I_T$. If $\{x_i \mid i \in I\}$ is a basis of L ordered by some indexing set I, then the set $\{x_{i_1} \cdots x_{i_n} \mid i_1 \leq \cdots \leq i_n\}$ forms a basis for the space T/I_T. This is known as the *Poincaré-Birkhoff-Witt* (PBW) Theorem. See for example [5, Section 17] for a proof of both the isomorphism $U \cong T/I_T$ and the PBW Theorem.

2.2. **Rational Case.** Let $\mu \in \mathbb{Q}^*$, $\mu = \frac{n}{m}$ with $\gcd(m, n) = 1$. Then for any indecomposable representation V of L_μ, β will act on V with eigenvalues from a set of the form $\gamma + \mathbb{Z}$ for some $\gamma \in \mathbb{C}$. We will write V_λ to represent the eigenspace of β with eigenvalue λ. This gives the following eigenspace decomposition:

$$V = \bigoplus\nolimits_{k \in \mathbb{Z}} V_{\gamma+k}, \qquad V_\lambda = \{v \in V \mid \beta \cdot v = \lambda v\}.$$

Let $U_{m,n}$ be the universal enveloping algebra of L_μ and let U^0, U^1, U^2 be the subalgebras generated by $\beta, \alpha_1, \alpha_2$ respectively. Then, by the PBW Theorem, we have

$$U_{m,n} \cong U^1 \otimes U^0 \otimes U^2 \quad \text{(as vector spaces)}. \tag{2.2.1}$$

From (1.2.3) we obtain the following relations in the universal enveloping algebra:

$$\beta\alpha_1 - \alpha_1\beta = m\alpha_1, \qquad \beta\alpha_2 - \alpha_2\beta = n\alpha_2, \qquad \alpha_1\alpha_2 = \alpha_2\alpha_1. \tag{2.2.2}$$

Let $x \in V_\lambda$. Then $\beta x = \lambda x$ so we have:

$$\begin{aligned} \beta(\alpha_1 x) &= (\beta\alpha_1)x \\ &= (m\alpha_1 + \alpha_1\beta)x \qquad \text{by (2.2.2)} \\ &= (\lambda + m)(\alpha_1 x) \\ &\Rightarrow \alpha_1 V_\lambda \subseteq V_{m+\lambda}. \end{aligned}$$

Similarly, we find that $\alpha_2 V_\lambda \subseteq V_{n+\lambda}$.

Following [8, Section 2] we will consider the modified enveloping algebra $\widetilde{U}_{m,n}$ of L_μ by replacing U^0 with a sum of 1-dimensional algebras:

$$\widetilde{U}_{m,n} = U^1 \otimes \left(\bigoplus\nolimits_{k \in \mathbb{Z}} \mathbb{C}a_k\right) \otimes U^2. \tag{2.2.3}$$

Multiplication in the modified enveloping algebra is given by

$$\begin{aligned} a_k a_\ell &= \delta_{k\ell} a_k, \\ \alpha_1 a_k = a_{k+m}\alpha_1, &\quad \alpha_2 a_k = a_{k+n}\alpha_2, \\ \alpha_1\alpha_2 a_k &= \alpha_2\alpha_1 a_k, \end{aligned} \tag{2.2.4}$$

where $k, \ell \in \mathbb{Z}$. We can think of a_k as the projection onto the k-th weight space $V_{\gamma+k}$. For any associative algebra A, we denote the category of representations of A by rep(A). If we denote by wt-rep$(U_{m,n})$ the category of representations of $U_{m,n}$ having weight space decomposition, then we have the equivalence of categories wt-rep$(U_{m,n}) \cong$ rep$(\widetilde{U}_{m,n})$.

2.3. **Non-Rational Case.** Using the same notation as in Section 2.2, we have V a representation of L_μ and V_λ the eigenspace of β with eigenvalue λ. However, when $\mu \in \mathbb{C}\setminus\mathbb{Q}$, β will act on indecomposable representations of L_μ with eigenvalues of the form $\gamma + k$ for some $\gamma \in \mathbb{C}$, where $k \in \mathbb{Z} + \mathbb{Z}\mu$. Therefore, we have $V = \bigoplus_{k\in\mathbb{Z}+\mathbb{Z}\mu} V_{\gamma+k}$.

When $\mu \in \mathbb{C}\setminus\mathbb{Q}$, we get the same decomposition of the universal enveloping algebra as found in (2.2.1). However, the relations found in (2.2.2) become

$$\beta\alpha_1 - \alpha_1\beta = \alpha_1, \qquad \beta\alpha_2 - \alpha_2\beta = \mu\alpha_2, \qquad \alpha_1\alpha_2 = \alpha_2\alpha_1. \tag{2.3.1}$$

As in the rational case, we find that $\alpha_1 V_\lambda \subseteq V_{\lambda+1}$ and $\alpha_2 V_\lambda \subseteq V_{\lambda+\mu}$.

Again we consider the modified enveloping algebra, $\widetilde{U}_\mu$, in this case given by

$$\widetilde{U}_\mu = U^1 \otimes \left(\bigoplus_{k\in\mathbb{Z}+\mu\mathbb{Z}} \mathbb{C}a_k\right) \otimes U^2. \tag{2.3.2}$$

Multiplication in this modified enveloping algebra is given by

$$\begin{aligned} a_k a_\ell &= \delta_{k\ell} a_k, \\ \alpha_1 a_k = a_{k+1}\alpha_1, &\quad \alpha_2 a_k = a_{k+\mu}\alpha_2, \\ \alpha_1\alpha_2 a_k &= \alpha_2\alpha_1 a_k, \end{aligned} \tag{2.3.3}$$

where $k, \ell \in \mathbb{Z} + \mu\mathbb{Z}$. Since $\mu \in \mathbb{C}\setminus\mathbb{Q}$, we have $\mathbb{Z}+\mu\mathbb{Z} \cong \mathbb{Z}\times\mathbb{Z}$, so we can reindex the projections a_k by defining $a_{ij} = a_{i+j\mu}$. In this notation the modified enveloping algebra has the form

$$\widetilde{U}_\mu = U^1 \otimes \left(\bigoplus_{i,j\in\mathbb{Z}} \mathbb{C}a_{ij}\right) \otimes U^2. \tag{2.3.4}$$

The multiplication is given by:

$$\begin{aligned} a_{ij}a_{st} &= \begin{cases} a_{ij}, & \text{if } i = s, j = t, \\ 0, & \text{otherwise,} \end{cases} \\ \alpha_1 a_{ij} = a_{(i+1)j}\alpha_1, &\quad \alpha_2 a_{ij} = a_{i(j+1)}\alpha_2, \\ \alpha_1\alpha_2 a_{ij} &= \alpha_2\alpha_1 a_{ij}. \end{aligned} \tag{2.3.5}$$

Remark 2.1. *The importance of this last point is that we have eliminated any dependence on μ. This shows that when $\mu \in \mathbb{C}\setminus\mathbb{Q}$ the modified enveloping algebras of all the Lie algebras L_μ are isomorphic. Thus from now on we will denote $\widetilde{U}_\mu$ simply by $\widetilde{U}$ when μ is not rational.*

Again we have an equivalence of categories wt-rep$(U_\mu) \cong$ rep$(\widetilde{U})$, where U_μ is the universal enveloping algebra of L_μ. Note that the categories of weight representations of L_μ are all equivalent whenever $\mu \in \mathbb{C}\setminus\mathbb{Q}$.

3. The Quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$

3.1. **Quivers and the Path Algebra.** A *quiver* Q is a 4-tuple (X, A, t, h), where X and A are sets, and t and h are functions from A to X. The sets X and A are called the vertex and arrow sets respectively. If $\rho \in A$, we call $t(\rho)$ the *tail* of ρ, and $h(\rho)$ the *head*. We can think of an element $\rho \in A$ as an arrow from the vertex $t(\rho)$ to the vertex $h(\rho)$. We will often denote a quiver simply by $Q = (X, A)$.

A *path* in a quiver Q is a sequence $\tau = \rho_n \rho_{n-1} \cdots \rho_1$ of arrows such that $h(\rho_i) = t(\rho_{i+1})$ for each $1 \le i \le n-1$. We define $t(\tau) = t(\rho_1)$ and $h(\tau) = h(\rho_n)$. For any quiver Q, we define its associated *path algebra* $\mathbb{C}Q$ to be the $\mathbb{C}$-algebra whose underlying vector space has for basis the set of paths in Q and with multiplication given by:

$$\tau_2 \cdot \tau_1 = \begin{cases} \tau_2\tau_1, & \text{if } h(\tau_1) = t(\tau_2), \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau_2\tau_1$ denotes the concatenation of the paths τ_1 and τ_2. In the following two sections we will construct isomorphisms between the modified enveloping algebras $\widetilde{U}_{m,n}$ and $\widetilde{U}$ found in Section 2 and the path algebras of certain quivers.

3.2. **Rational Case.** Let $\mu = \frac{n}{m}$, where $m, n \in \mathbb{Z}$ and $\gcd(m, n) = 1$. We will consider the quiver $Q_{m,n} = (\mathbb{Z}, A^{m,n})$ where $A^{m,n} = \{\rho_j^k \mid k \in \mathbb{Z},\ j = 1, 2\}$ is the arrow set. We define a map $\sigma : \{1, 2\} \to \{m, n\}$ such that $\sigma(1) = m$ and $\sigma(2) = n$. Then ρ_j^k is an arrow whose tail, $t(\rho_j^k)$, is the vertex k and whose head, $h(\rho_j^k)$, is the vertex $k + \sigma(j)$.

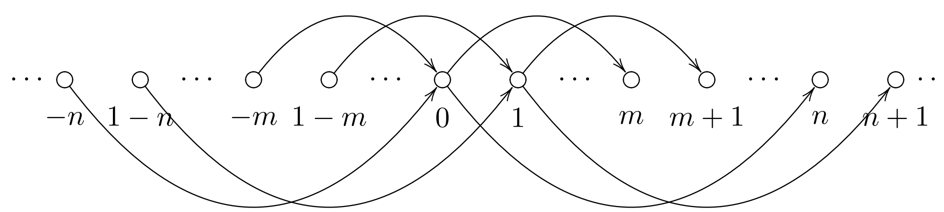

Figure 1. The Quiver $Q_{m,n}$

Any path in the quiver $Q_{m,n}$ is of the form $\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_2}^{k+\sigma(j_1)} \rho_{j_1}^k$ with $k \in \mathbb{Z}$, $s \in \mathbb{N}$ and $j_i \in \{1, 2\}$ for $i = 1, \ldots, s$. Elements of the path algebra $\mathbb{C}Q_{m,n}$ are linear combinations of these paths.

Consider the linear map φ determined by:

$$\begin{aligned} \varphi : \mathbb{C}Q_{m,n} &\to \tilde{U}, \\ \rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k &\mapsto \alpha_{j_s} \cdots \alpha_{j_1} a_k. \end{aligned} \tag{3.2.1}$$

Lemma 3.1. *The map φ is a homomorphism of algebras.*

Proof. Since φ is linear, it suffices to show that it commutes with the multiplication. Let

$$\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k, \rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})} \cdots \rho_{i_1}^l \in \mathbb{C}Q_{m,n}.$$

Then

$$
\begin{aligned}
&(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k})\cdot(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l})\\
&\quad=\begin{cases}\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l} & \text{if } l+\sigma(i_1)+\cdots+\sigma(i_r)=k\\ 0 & \text{otherwise}\end{cases}\\
\Longrightarrow\ &\varphi\left((\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k})\cdot(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l})\right)\\
&\quad=\begin{cases}\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\cdots\alpha_{i_1}a_l & \text{if } l+\sigma(i_1)+\cdots+\sigma(i_r)=k,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\varphi\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}\right)\varphi\left(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l}\right)\\
&\quad=(\alpha_{j_s}\cdots\alpha_{j_1}a_k)(\alpha_{i_r}\cdots\alpha_{i_1}a_l)\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}a_{k-\sigma(i_r)}\alpha_{i_{r-1}}\cdots\alpha_{i_1}a_l\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\alpha_{i_{r-1}}a_{k-\sigma(i_r)-\sigma(i_{r-1})}\alpha_{i_{r-2}}\cdots\alpha_{i_1}a_l\\
&\quad\ \vdots\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\cdots\alpha_{i_1}a_{k-\sigma(i_r)-\cdots-\sigma(i_1)}a_l\\
&\quad=\begin{cases}\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\cdots\alpha_{i_1}a_l & \text{if } l=k-\sigma(i_1)-\cdots-\sigma(i_r),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

And hence

$$
\begin{aligned}
&\varphi\left((\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k})\cdot(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l})\right)\\
&\quad=\varphi\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}\right)\varphi\left(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l}\right).
\end{aligned}
$$

□

We now establish a relationship between the path algebra $\mathbb{C}Q_{m,n}$ and the modified enveloping algebra $\widetilde{U}_{m,n}$.

Proposition 3.2. *For any $m,n\in\mathbb{Z}^*$ there is an isomorphism of algebras $\widetilde{U}_{m,n}\cong\mathbb{C}Q_{m,n}/I^{m,n}$, where $I^{m,n}$ is the two-sided ideal generated by elements of the form $\rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k$.*

Proof. We claim that $I^{m,n}\subseteq\ker\varphi$. Indeed:

$$
\begin{aligned}
\varphi(\rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k)&=\varphi(\rho_1^{k+n}\rho_2^k)-\varphi(\rho_2^{k+m}\rho_1^k)\\
&=\alpha_1\alpha_2a_k-\alpha_2\alpha_1a_k\\
&=\alpha_1\alpha_2a_k-\alpha_1\alpha_2a_k \qquad \text{(by (2.2.4))}\\
&=0\\
\Rightarrow \rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k&\in\ker\varphi\quad\forall k\in\mathbb{Z}.
\end{aligned}
$$

Thus φ induces a morphism

$$
\begin{aligned}
\bar{\varphi}:\mathbb{C}Q_{m,n}/I^{m,n}&\to\tilde{U},\\
x+I&\mapsto\varphi(x)\qquad\forall x\in\mathbb{C}Q_{m,n}.
\end{aligned}
$$

We will also consider the linear map determined by:

$$
\begin{aligned}
\psi:\widetilde{U}&\to\mathbb{C}Q_{m,n}/I^{m,n}\\
{\alpha_1}^r a_k{\alpha_2}^s&\mapsto\rho_1^{k+(r-1)m}\cdots\rho_1^k\rho_2^{k-n}\cdots\rho_2^{k-sn}+I.
\end{aligned}
$$

We will show that $\psi\bar{\varphi}$ and $\bar{\varphi}\psi$ are identity maps. Seeing as both maps are linear, it will suffice to show that this is the case for basis elements. Let $\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}+I\in\mathbb{C}Q/I^{m,n}$. Then

$$\begin{aligned}(\psi\bar{\varphi})\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}+I\right) &= \psi\left(\bar{\varphi}(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}+I)\right)\\ &= \psi\left(\varphi(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k})\right)\\ &= \psi(\alpha_{j_s}\cdots\alpha_{j_1}a_k)\\ &= \psi(\alpha_1^r\alpha_2^t a_k), \quad \text{for some } r+t=s\\ &= \psi(\alpha_1^r a_{k+tn}\alpha_2^t)\\ &= \rho_1^{k+tn+(r-1)m}\cdots\rho_1^{k+tn}\rho_2^{k+(t-1)n}\cdots\rho_2^{k}+I\\ &= \rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}+I\end{aligned}$$

$$\implies \psi\bar{\varphi}=\mathrm{id}:\mathbb{C}Q_{m,n}/I^{m,n}\to\mathbb{C}Q_{m,n}/I^{m,n}.$$

In the last line of the computation we used the relation $\rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k\in I^{m,n}$ to reorder the terms into a new member of the same equivalence class.

Now let ${\alpha_1}^r a_k{\alpha_2}^s\in\widetilde{U}$. Then

$$\begin{aligned}(\bar{\varphi}\psi)({\alpha_1}^r a_k{\alpha_2}^s) &= \bar{\varphi}\left(\psi({\alpha_1}^r a_k{\alpha_2}^s)\right)\\ &= \bar{\varphi}\left(\rho_1^{k+(r-1)m}\cdots\rho_1^k\rho_2^{k-n}\cdots\rho_2^{k-sn}+I\right)\\ &= \varphi\left(\rho_1^{k+(r-1)m}\cdots\rho_1^k\rho_2^{k-n}\cdots\rho_2^{k-sn}\right)\\ &= {\alpha_1}^r{\alpha_2}^s a_{k-sn}\\ &= {\alpha_1}^r a_k{\alpha_2}^s\end{aligned}$$

$$\implies \bar{\varphi}\psi=\mathrm{id}:\widetilde{U}\to\widetilde{U}.$$

Combining this result with Lemma 3.1, we see that $\bar{\varphi}$ is a bijective homomorphism of algebras, which completes the proof. □

Thus there is an equivalence of categories $\mathrm{rep}(\widetilde{U}_{m,n})\cong\mathrm{rep}(\mathbb{C}Q_{m,n}/I^{m,n})$.

3.3. **Non-Rational Case.** We now consider the quiver $Q_{\infty\times\infty}=(\mathbb{Z}\times\mathbb{Z},A_{\infty\times\infty})$ where the set of arrows is $A_{\infty\times\infty}=\{\rho_d^k\mid d\in\{1,2\},k\in\mathbb{Z}\times\mathbb{Z}\}$. We define the map $\theta:\{1,2\}\to\{(1,0),(0,1)\}$ by $\theta(1)=(1,0)$ and $\theta(2)=(0,1)$. Then ρ_d^k is the arrow whose tail, $t(\rho_d^k)$, is the vertex $k=(i,j)$ and whose head, $h(\rho_d^k)$, is the vertex $(i,j)+\theta(d)$. Elements of the form $\rho_{d_s}^{k+\theta(d_1)+\cdots+\theta(d_{s-1})}\cdots\rho_{d_2}^{k+\theta(d_1)}\rho_{d_1}^k$ constitute a basis for the path algebra $\mathbb{C}Q_{\infty\times\infty}$, where $k\in\mathbb{Z}\times\mathbb{Z}$, $s\in\mathbb{N}$, and $d_n\in\{1,2\}$ for $n=1,...,s$.

Consider the linear map Ω defined by:

$$\begin{aligned}\Omega:\mathbb{C}Q_{\infty\times\infty}&\to\widetilde{U},\\ \rho_{d_s}^{k+\theta(d_1)+\cdots+\theta(d_{s-1})}\cdots\rho_{d_1}^{k}&\mapsto\alpha_{d_s}\cdots\alpha_{d_1}a_{ij},\end{aligned}\tag{3.3.1}$$

where $k=(i,j)$.

Lemma 3.3. *The map Ω is a homomorphism of algebras.*

Proof. The proof is similar to that of Lemma 3.1. □

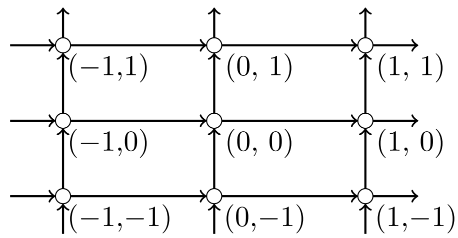

FIGURE 2. The Quiver $Q_{\infty\times\infty}$

Proposition 3.4. *There is an isomorphism of algebras* $\widetilde{U} \cong \mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}$ *where* $I_{\infty\times\infty}$ *is the two-sided ideal generated by elements of the form* $\rho_1^{k+(0,1)}\rho_2^k - \rho_2^{k+(1,0)}\rho_1^k$*, where* $k \in \mathbb{Z}\times\mathbb{Z}$*.*

Proof. The proof is similar to that of Proposition 3.2. □

So we have an equivalence of categories $\mathrm{rep}(\widetilde{U}) \cong \mathrm{rep}(\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty})$.

4. Representations

4.1. **Quiver Representations.** If X is a set then we denote by $\mathcal{V}(X)$ the category of X-graded vector spaces over $\mathbb{C}$. A *representation* of a quiver $Q = (X, A)$ is an element $V \in \mathcal{V}(X)$ along with a collection of linear maps $\{v_\rho : V_{t(\rho)} \to V_{h(\rho)} \mid \rho \in A\}$. Let $V = (V_i, v_\rho), W = (W_i, w_\rho)$ be two representations of a quiver $Q = (X, A)$. A *morphism* from V to W, $x \in \mathrm{Hom}(V, W)$, is a collection of linear maps $\{x_i : V_i \to W_i\}$ such that for every $\rho \in A$ the following diagram commutes:

$$\begin{array}{ccc} W_{t(\rho)} & \xrightarrow{w_\rho} & W_{h(\rho)} \\ {\scriptstyle x_{t(\rho)}}\uparrow & & \uparrow{\scriptstyle x_{h(\rho)}} \\ V_{t(\rho)} & \xrightarrow{v_\rho} & V_{h(\rho)} \end{array} \tag{4.1.1}$$

Thus we may consider the category $\mathrm{rep}(Q)$ having as objects representations of Q, and with morphisms as described above. The category of representations of the quiver $Q = (X, A)$ is equivalent to the category of representations of the path algebra $\mathbb{C}Q$. In particular, there is a 1-1 correspondance between representations of Q and representations of $\mathbb{C}Q$; for any path $\tau = \rho_n\rho_{n-1}\cdots\rho_1$, we define $x_\tau = x_{\rho_n}x_{\rho_{n-1}}\cdots x_{\rho_1}$. We say that a representation *satisfies the relation* $\sum_{j=1}^k a_j\rho_j$, where $a_j \in \mathbb{C}$ and $\rho_j \in A$ for $j \in \{1, \dots, k\}$, if $\sum_{j=1}^k a_j x_j = 0$. If R is a set of relations, we denote by $\mathrm{rep}(Q, R)$ the category of representations of Q which satisfy the relations R. If I is the two-sided ideal generated by the relations R, then there is an equivalence of categories $\mathrm{rep}(\mathbb{C}Q/I) \cong \mathrm{rep}(Q, R)$.

Remark 4.1. *If A is an algebra then there is a natural equivalence of categories* $\mathrm{rep}(A) \cong A$*-Mod. Thus the equivalences described above imply that the categories* $\mathrm{rep}(Q)$ *and* $\mathrm{rep}(Q, R)$ *are abelian. For more information on categories of representations of quivers, see* [1].

Let $Q = (X, A)$ be some quiver. If $V, U \in \mathrm{rep}(Q)$ and $\varphi \in \mathrm{Hom}(V, U)$, then the *kernel* of φ, $\ker\varphi$, is the representation of Q with vector spaces given by $(\ker\varphi)_i = \ker(\varphi_i)$, $\forall i \in X$, and maps given by the restriction of the maps in V to these subspaces. The *cokernel* of φ is the representation of Q given by $(\mathrm{coker}\,\varphi)_i = (U_i/\mathrm{im}\,\varphi_i)$ and with maps induced on these spaces by the maps of U.

We conclude this section by recalling some properties of functors which will be used later on. Let $\mathcal{C}$ and $\mathcal{A}$ be abelian categories, and let $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ be a functor. Then for any $A, B \in \mathcal{C}$, the functor $\mathcal{F}$ induces a map $\mathcal{F}_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$. If $\mathcal{F}_{AB}$ is a group homomorphism for all A, B, we say $\mathcal{F}$ is *additive*. If $\mathcal{F}_{AB}$ is injective (resp. surjective) for all A, B, we say $\mathcal{F}$ is *faithful* (resp. *full*). If $\mathcal{F}$ is additive, then it preserves finite coproducts; $\mathcal{F}(A \coprod B) \cong \mathcal{F}(A) \coprod \mathcal{F}(B)$, see for example [7, Corollary 5.88] for a proof of this statement. If for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, the sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$ is exact, then we say the functor $\mathcal{F}$ is *exact*. A functor between abelian categories is exact if and only if it preserves kernels and cokernels.

4.2. **Rational Case.** Let $\mu = \frac{n}{m}$, $\gcd(m, n) = 1$. We have the equivalences

$$\mathrm{rep}(\widetilde{U}_{m,n}) \cong \mathrm{rep}(\mathbb{C}Q_{m,n}/I^{m,n}) \cong \mathrm{rep}(Q_{m,n}, R^{m,n}),$$

where $R^{m,n} = \{\rho_1^{k+n}\rho_2^k - \rho_2^{k+m}\rho_1^k \mid k \in \mathbb{Z}\}$. We define $\widehat{Q}_t = (\mathbb{Z}/t\mathbb{Z}, \rho_i, \bar{\rho}_i)$ for some $t \in \mathbb{N}$, where $t(\rho_i) = h(\bar{\rho}_{i+1}) = i$ and $h(\rho_i) = t(\bar{\rho}_{i+1}) = i + 1$. If we also define $\widehat{R} = \{\bar{\rho}_{i+1}\rho_i - \rho_{i-1}\bar{\rho}_i \mid i \in \mathbb{Z}/t\mathbb{Z}\}$, then we can study the representations of $\widetilde{U}_{m,n}$ by relating the category $\mathrm{rep}(Q_{m,n}, R^{m,n})$ to the category $\mathrm{rep}(\widehat{Q}_{m+n}, \widehat{R})$. This is an interesting connection, and moreover much is known about the category $\mathrm{rep}(\widehat{Q}_{m+n}, \widehat{R})$, see [3] for example.

Let $V = (V_k, x_i^k) \in \mathrm{rep}(Q_{m,n})$ and let $j \in \mathbb{Z}/(m+n)\mathbb{Z}$. If $k \in \mathbb{Z}$, we will write $k \equiv j \mod (m+n)$ simply as $k \equiv j$. We define new vector spaces V^j by

$$V^j = \bigoplus_{k \equiv jm} V_k.$$

We also define linear maps between these spaces: $x^j = \bigoplus_{k\equiv jm} x_1^k$ and $\bar{x}^j = \bigoplus_{k\equiv jm} x_2^k$. Note that x^j maps V^j to V^{j+1}, and $\bar{x}^j$ maps V^j to V^{j-1}. Thus we have defined a map which takes a representation $V = (V_k, x_1^k, x_2^k)$ of $Q_{m,n}$ to the representation $V' = (V^j, x^j, \bar{x}^j)$ of $\widehat{Q}_{m+n}$. We denote this map by $\mathcal{G}$. We can extend $\mathcal{G}$ to act on morphisms $f = \{f_k : V_k \to U_k\}$ between two representations V, U of $Q_{m,n}$ by defining $\mathcal{G}(f) = \{f^j : V^j \to U^j\}$, where $f^j = \bigoplus_{k\equiv jm} f_k$. It is clear that $\mathcal{G}(f) \in \mathrm{Hom}(\mathcal{G}(V), \mathcal{G}(U))$. We claim that $\mathcal{G}$ preserves the identity morphism and also preserves compositions of morphisms, and hence defines a *functor*. Indeed, if $f \in \mathrm{Hom}(V, V)$ is the identity morphism, then $f_k = \mathrm{id} : V_k \to V_k\ \forall k$. Hence $f^j = \mathrm{id} : V^j \to V^j\ \forall j$. It follows that $\mathcal{G}(f) = \mathrm{id} : \mathcal{G}(V) \to \mathcal{G}(V)$. If $f : V \to U$ and $g : U \to W$ are two morphisms then their composition is the morphism $gf : V \to W$ defined by $(gf)_k = g_k f_k$ for all k. Then $(gf)^j = \bigoplus_{k\equiv jm}(gf)_k = \bigoplus_{k\equiv jm} g_k f_k = (\bigoplus_{k\equiv jm} g_k)(\bigoplus_{k\equiv jm} f_k) = g^j f^j$. Since this is true for every j, we must have $\mathcal{G}(gf) = \mathcal{G}(g)\mathcal{G}(f)$, as claimed. We now establish certain properties of the functor $\mathcal{G}$.

Lemma 4.2. *The functor $\mathcal{G}$ is additive.*

Proof. Denote by $\mathcal{G}_{VU}$ the restriction of the functor $\mathcal{G}$ to the space $\mathrm{Hom}(V, U)$, $\mathcal{G}_{VU} : \mathrm{Hom}(V, U) \to \mathrm{Hom}(\mathcal{G}(V), \mathcal{G}(U))$. If $f, g \in \mathrm{Hom}(V, U)$, then$(f + g)^j = \bigoplus_{k\equiv jm}(f + g)_k = \bigoplus_{k\equiv jm}(f_k + g_k) = (\bigoplus_{k\equiv jm} f_k) + (\bigoplus_{k\equiv jm} g_k) = f^j + g^j$. Since this is true for every j, we have $\mathcal{G}_{VU}(f + g) = \mathcal{G}_{VU}(f) + \mathcal{G}_{VU}(g)$. Thus $\mathcal{G}_{VU}$ is a group homomorphism, so $\mathcal{G}$ is an additive functor. □

Let $V, U \in \mathrm{rep}(Q_{m,n})$. Then $\mathcal{G}(V) \ncong \mathcal{G}(U) \Rightarrow V \ncong U$ since $\mathcal{G}$ is a functor. Further, it follows from Lemma 4.2 that if $\mathcal{G}(V)$ is indecomposable then V is indecomposable, since additive functors preserve finite coproducts, which in the categories $\mathrm{rep}(Q_{m,n})$ and $\mathrm{rep}(\widehat{Q}_{m+n})$ are finite direct sums.

Lemma 4.3. *The functor $\mathcal{G}$ is faithful.*

Proof. Let $V, U \in \mathrm{rep}(Q_{m,n})$ and consider again the homomorphism $\mathcal{G}_{VU} : \mathrm{Hom}(V, U) \to \mathrm{Hom}(\mathcal{G}(V), \mathcal{G}(U))$. We wish to show that this map is injective. Since it is a group homomorphism, it is enough to consider the preimage of the morphism $0 : \mathcal{G}(V) \to \mathcal{G}(U)$. This morphism is defined by the linear maps $0^j = 0 : V^j \to U^j$. So if $\mathcal{G}(f) = 0$, then we have $\bigoplus_{k\equiv jm} f_k = 0$ for every j, and it follows that $f_k = 0$ for all k. Hence $f = 0$. □

Another important property that the functor $\mathcal{G}$ possesses is exactness.

Lemma 4.4. *The functor $\mathcal{G}$ is exact.*

Proof. Since the categories $\operatorname{rep}(Q_{m,n})$ and $\operatorname{rep}(\widehat{Q}_{m+n})$ are abelian, it is enough to show that $\mathcal{G}$ preserves kernels and cokernels. Let $V, U \in \operatorname{rep}(Q_{m,n})$, and let $\varphi \in \operatorname{Hom}(V, U)$. Then we have $\ker(\mathcal{G}(\varphi))^j = \ker(\bigoplus_{k\equiv jm} \varphi_k) = \bigoplus_{k\equiv jm} \ker \varphi_k$ for all $j \in \mathbb{Z}/(m+n)\mathbb{Z}$, and it follows $\ker \mathcal{G}(\varphi) = \mathcal{G}(\ker \varphi)$ so $\mathcal{G}$ preserves kernels. We also have $\operatorname{im}(\mathcal{G}(\varphi))^j = \operatorname{im}(\bigoplus_{k\equiv jm} \varphi_k) = \bigoplus_{k\equiv jm} \operatorname{im} \varphi_k$, so $\mathcal{G}(U)_j / \operatorname{im} \mathcal{G}(\varphi)_j = (\bigoplus_{k\equiv jm} U_k)/(\bigoplus_{k\equiv jm} \operatorname{im} \varphi_k) = \bigoplus_{k\equiv jm}(U_k/\operatorname{im} \varphi_k)$ for every $j \in \mathbb{Z}/(m+n)\mathbb{Z}$. Hence $\operatorname{coker} \mathcal{G}(\varphi) = \mathcal{G}(\operatorname{coker} \varphi)$, so $\mathcal{G}$ preserves cokernels. □

Let $V \in \operatorname{rep}(Q_{m,n}, R^{m,n})$. Then since $x_1^{k+n} x_2^k - x_2^{k+m} x_1^k = 0 \ \forall k \in \mathbb{Z}$, we get:

$$\begin{aligned}
\bar{x}^{j+1} x^j - x^{j-1} \bar{x}^j &= \left(\bigoplus_{k\equiv(j+1)m} x_2^k\right)\left(\bigoplus_{k\equiv jm} x_1^k\right) - \left(\bigoplus_{k\equiv(j-1)m} x_1^k\right)\left(\bigoplus_{k\equiv jm} x_2^k\right) \\
&= \bigoplus_{k\equiv jm} x_2^{k+m} x_1^k - \bigoplus_{k\equiv jm} x_1^{k-m} x_2^k \\
&= \bigoplus_{k\equiv jm} \left(x_2^{k+m} x_1^k - x_1^{k+n} x_2^k\right) \\
&= 0.
\end{aligned}$$

Hence we can restrict the functor $\mathcal{G}$ to the subcategory $\operatorname{rep}(Q_{m,n}, R^{m,n})$ of $\operatorname{rep}(Q_{m,n})$ to get a functor $\bar{\mathcal{G}} : \operatorname{rep}(Q_{m,n}, R^{m,n}) \to \operatorname{rep}(\widehat{Q}_{m+n}, \widehat{R})$. Further, all of the properties that were proven for $\mathcal{G}$ still hold for the restricted functor $\bar{\mathcal{G}}$, and we can therefore relate the categories $\operatorname{rep}(\widetilde{U}_{m,n})$ and $\operatorname{rep}(\widehat{Q}_{m+n}, \widehat{R})$. It is natural to ask whether or not $\bar{\mathcal{G}}$ gives an equivalence of categories, and it turns out that this is true only when $\mu = -1$, in which case $\mathcal{G}$ is the identity functor. When $\mu \neq -1$, the functor $\bar{\mathcal{G}}$ is neither full nor essentially surjective, as the following examples illustrate:

Example 4.5. Let $V \in \operatorname{rep}(Q_{m,n}, R^{m,n})$ be the representation given by $V_0 = V_{n+m} = \mathbb{C}$, and $V_i = 0$ for all other $i \in \mathbb{Z}$. The endomorphism space of V is $\operatorname{Hom}(V, V) \cong \operatorname{Hom}(\mathbb{C}, \mathbb{C}) \oplus \operatorname{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}^2$. The representation $\bar{\mathcal{G}}(V) \in \operatorname{rep}(\widehat{Q}_{m+n}, \widehat{R})$ is the representation such that $\bar{\mathcal{G}}(V)_0 = \mathbb{C}^2$, and all other vector spaces are zero. Thus $\operatorname{Hom}(\bar{\mathcal{G}}(V), \bar{\mathcal{G}}(V)) \cong \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^2) \cong \operatorname{Mat}_{2\times 2}(\mathbb{C})$. Since the dimension of the endomorphism space of $\bar{\mathcal{G}}(V)$ is greater than the dimension of the endomorphism space of V, the map $\bar{\mathcal{G}}_{VV}$ is not surjective. Hence $\bar{\mathcal{G}}$ is not full.

Example 4.6. Let $U \in \operatorname{rep}(Q_{m,n}, R^{m,n})$. Then if U is finite dimensional, there are only finitely many nonzero U_k, and so there exists some $t \in \mathbb{Z}$ such that $x_1^{k+tm} x_1^{k+(t-1)m} \cdots x_1^k = 0$ for any $k \in \mathbb{Z}$. Then $\bigoplus_{k\equiv jm} x_1^{k+tm} x_1^{k+(t-1)m} \cdots x_1^k = 0$, and so there is path in the representation $\bar{\mathcal{G}}(U) \in \operatorname{rep}(\widehat{Q}_{m+n}, \widehat{R})$ that acts by zero. Consider the representation $V \in \operatorname{rep}(\widehat{Q}_{m+n}, \widehat{R})$ defined by $(V_i, x_i, \bar{x}_i) = (\mathbb{C}, \lambda, 1)$ for all $i \in \mathbb{Z}/(m+n)\mathbb{Z}$, where $\lambda \in \mathbb{C}$. Suppose there were some representation $U \in \operatorname{rep}(Q_{m,n}, R^{m,n})$ such that $\bar{\mathcal{G}}(U) \cong V$. Then since V is finite dimensional, U must also be finite dimensional. However, since there are no paths in the representation V that act by zero, this is a contradiction. Hence $\bar{\mathcal{G}}$ is not essentially surjective.

Let $Q = (X, A)$ be a quiver. For every $\rho \in A$, define an arrow $\bar{\rho}$ by $t(\bar{\rho}) = h(\rho)$ and $h(\bar{\rho}) = t(\rho)$, and let $\bar{A}$ denote the set of all such $\bar{\rho}$. Then we call $\bar{Q} = (X, A \cup \bar{A})$ the *double quiver* of Q. Let I be the two sided ideal of $\mathbb{C}\bar{Q}$ generated by elements of the form

$$\sum_{\rho\in A, h(\rho)=i} \rho\bar{\rho} - \sum_{\rho\in A, t(\rho)=i} \bar{\rho}\rho.$$

Then the algebra $\mathbb{C}\bar{Q}/I$ is called the *preprojective algebra* of Q, and is denoted $\mathcal{P}(Q)$. If $\widehat{I}$ denotes the two sided ideal of $\mathbb{C}\widehat{Q}_{m+n}$ generated by the relations $\widehat{R}$, then we have $\mathbb{C}\widehat{Q}_{m+n}/\widehat{I} = \mathcal{P}(\widehat{Q}^*_{m+n})$, where $\widehat{Q}^*_{m+n}$ is a subquiver of $\widehat{Q}_{m+n}$ obtained by specifying an orientation. The functor $\bar{\mathcal{G}}$ provides an embedding of the category $\mathrm{rep}(\mathbb{C}^{m,n}/I^{m,n})$ in $\mathrm{rep}(\mathcal{P}(\widehat{Q}^*_{m+n}))$.

Note that if a representation $V \in \mathrm{rep}(Q_{m+n}, R^{m,n})$ is finite dimensional, then $\bar{\mathcal{G}}(V)$ will be *nilpotent*. For any preprojective algebra $\mathcal{P}(Q)$, the set of nilpotent representations of $\mathcal{P}(Q)$ forms a variety called *Lusztig's quiver variety* (see [6]), and is denoted Λ_Q. Thus the functor $\bar{\mathcal{G}}$ embeds the finite dimensional objects of the category $\mathrm{rep}(\mathbb{C}Q_{\infty\times\infty}/I^{m,n})$ inside $\Lambda_{\widehat{Q}^*_{m+n}}$, and so we can use the variety $\Lambda_{\widehat{Q}^*_{m+n}}$ to study the representation theory of L_μ when $\mu = \frac{n}{m}$.

4.3. **Non-Rational Case.** When $\mu \in \mathbb{C} \setminus \mathbb{Q}$ we are interested in representations of the quiver $Q_{\infty\times\infty}$ introduced in Section 3.3. Then we have the equivalences $\mathrm{rep}(\widetilde{U}) \cong \mathrm{rep}(\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}) \cong \mathrm{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$, where $R_{\infty\times\infty} = \{\rho_1^{k+(0,1)}\rho_2^k - \rho_2^{k+(1,0)}\rho_1^k \mid k \in \mathbb{Z}\times\mathbb{Z}\}$. In order to study the representations of $Q_{\infty\times\infty}$ we will relate the category $\mathrm{rep}(Q_{\infty\times\infty})$ to the category $\mathrm{rep}(Q_\infty)$. Here Q_∞ denotes the quiver $(\mathbb{Z}, \rho_i, \bar{\rho}_i)$, where $t(\rho_i) = i = h(\bar{\rho}_{i+1})$, and $h(\rho_i) = i+1 = t(\bar{\rho}_{i+1})$. We will then be able to relate the category $\mathrm{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ to the category $\mathrm{rep}(Q_\infty, R_\infty)$, where $R_\infty = \{\bar{\rho}_{i+1}\rho_i - \rho_{i-1}\bar{\rho}_i \mid i \in \mathbb{Z}\}$. We will obtain a relationship between these two categories which is similar to the relationship we studied in Section 4.2. The representation theory of the quiver Q_∞ subject to relations R_∞ is well known, see [8] for a summary of the results.

Let (V_{ij}, x_{ij}, y_{ij}) be a representation of the quiver $Q_{\infty\times\infty}$, where $x_{ij} : V_{ij} \to V_{(i+1)j}$ and $y_{ij} : V_{ij} \to V_{i(+1)}$. We construct new vector spaces out of this representation by taking the direct sum of all vector spaces which lie along the same diagonal of the lattice. More formally, we define the vector spaces V^k by

$$V^k := \bigoplus_{i-j=k} V_{ij}.$$

Similarly, we define linear maps between these spaces by $x^k = \bigoplus_{i-j=k} x_{ij}$ and $y^k = \bigoplus_{i-j=k} y_{ij}$. Note that x^k maps V^k to V^{k-1} and y^k maps V^k to V^{k+1}. In other words, $\{(V^k, x^k, y^k) \mid k \in \mathbb{Z}\}$ constitutes a representation of Q_∞. Thus we have constructed a map of objects of $\mathrm{rep}(Q_{\infty\times\infty})$ to objects of $\mathrm{rep}(Q_\infty)$. We denote this map by $\mathcal{F}$. We can extend this map to act on morphisms of $\mathrm{rep}(Q_{\infty\times\infty})$ as follows: if $f = \{f_{ij}\}$ is a morphism between two representations V and U of $Q_{\infty\times\infty}$, where $f_{ij} : V_{ij} \to U_{ij}$, then we define $\mathcal{F}(f) = \{f^k\}$, where $f^k = \bigoplus_{i-j=k} f_{ij}$. It is easy to see that $\mathcal{F}(f)$ is a morphism from $\mathcal{F}(V)$ to $\mathcal{F}(U)$, and we claim that with these definitions $\mathcal{F}$ is a functor. Indeed, if f is the identity morphism, then for every i,j $f_{ij} = \mathrm{id} : V_{ij} \to V_{ij}$, and hence $f^k = \mathrm{id} : V^k \to V^k$ so that $\mathcal{F}$ preserves the identity morphism. If $f : V \to U$ and $g : U \to W$ are two morphisms then their composition is the morphism $gf : V \to W$ defined by $(gf)_{ij} = g_{ij} f_{ij}$ for all i,j. Then $(gf)^k = \bigoplus_{i-j=k}(gf)_{ij} = \bigoplus_{i-j=k} g_{ij} f_{ij} = (\bigoplus_{i-j=k} g_{ij})(\bigoplus_{i-j=k} f_{ij}) = g^k f^k$, and it follows that $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$.

Lemma 4.7. *The functor $\mathcal{F}$ is additive.*

Proof. Denote by $\mathcal{F}_{VU}$ the restriction of the functor $\mathcal{F}$ to $\mathrm{Hom}(V,U)$, $\mathcal{F}_{VU} : \mathrm{Hom}(V,U) \to \mathrm{Hom}(\mathcal{F}(V), \mathcal{F}(U))$. If $f, g \in \mathrm{Hom}(V,U)$, consider $(f+g)^k = \bigoplus_{i-j=k}(f+g)_{ij} = \bigoplus_{i-j=k}(f_{ij}+g_{ij}) = (\bigoplus_{i-j=k} f_i j) + (\bigoplus_{i-j=k} g_{ij}) = f^k + g^k$, and hence $\mathcal{F}_{VU}(f+g) = \mathcal{F}_{VU}(f) + \mathcal{F}_{VU}(g)$. Thus $\mathcal{F}_{VU}$ is a group homomorphism, so $\mathcal{F}$ is an additive functor. □

Thus if two objects $\mathcal{F}(V), \mathcal{F}(U) \in \mathrm{rep}(Q_\infty)$ are non-isomorphic, the objects $V, U \in \mathrm{rep}(Q_{\infty\times\infty})$ must be non-isomorphic. Also, if an object $\mathcal{F}(V) \in \mathrm{rep}(Q_\infty)$ is indecomposable, then the object $V \in \mathrm{rep}(Q_{\infty\times\infty})$ is also indecomposable, as in the rational case.

Lemma 4.8. *The functor $\mathcal{F}$ is faithful.*

Proof. Let $V, U \in \operatorname{rep}(Q_{\infty\times\infty})$ and let $f \in \operatorname{Hom}(V, U)$. Suppose $\mathcal{F}(f) = 0$. Then $f^k = \bigoplus_{i-j=k} f_{ij} = 0$ for every k, and hence $f_{ij} = 0$ for all i, j. But then $f = 0$, so the homomorphism $\mathcal{F}_{VU}$ is injective, as claimed. □

We can also show that $\mathcal{F}$ is exact.

Lemma 4.9. *The functor $\mathcal{F}$ is exact.*

Proof. Let $V, U \in \operatorname{rep}(Q_{\infty\times\infty})$, and let $\varphi \in \operatorname{Hom}(V, U)$. Then

$$\ker(\mathcal{F}(\varphi))^k = \ker(\textstyle\bigoplus_{i-j=k} \varphi_{ij}) = \bigoplus_{i-j=k} \ker \varphi_{ij},$$

so $\ker \mathcal{F}(\varphi) = \mathcal{F}(\ker \varphi)$. Thus $\mathcal{F}$ preserves kernels. Since $\operatorname{im}(\mathcal{F}(\varphi))^k = \bigoplus_{i-j=k} \operatorname{im} \varphi_{ij}$, we have that

$$\mathcal{F}(U)_k / \operatorname{im} \mathcal{F}(\varphi)_k = (\textstyle\bigoplus_{i-j=k} U_{ij})/(\bigoplus_{i-j=k} \operatorname{im} \varphi_{ij}) = \bigoplus_{i-j=k}(U_{ij}/\operatorname{im} \varphi_{ij}).$$

Thus $\operatorname{coker} \mathcal{F}(\varphi) = \mathcal{F}(\operatorname{coker} \varphi)$, so $\mathcal{F}$ preserves cokernels. □

The condition $y_{(i+1)j} x_{ij} - x_{i(j+1)} y_{ij} = 0$ for all i, j implies that $y^{k-1}x^k - x^{k+1}y^k = 0$ for all k. Thus $\mathcal{F}$ can be restricted to the subcategory $\operatorname{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ of $\operatorname{rep}(Q_{\infty\times\infty})$ to yield a functor $\bar{\mathcal{F}} : \operatorname{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty}) \to \operatorname{rep}(Q_\infty, R_\infty)$, and this restricted functor shares the properties proven for $\mathcal{F}$. However, the following examples illustrate that $\bar{\mathcal{F}}$ is neither full nor essentially surjective:

Example 4.10. Consider the following representation $V \in \operatorname{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0, \end{array}$$

where all vector spaces not shown in the diagram are zero. The endomorphism space of V is given by $\operatorname{Hom}(V, V) \cong \operatorname{Hom}(\mathbb{C}, \mathbb{C}) \oplus \operatorname{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}^2$. The object $\bar{\mathcal{F}}(V)$ is the representation of Q_∞ such that $\bar{\mathcal{F}}(V)_0 = \mathbb{C}^2$, and all other vertices are 0. The endomorphism space of this representation, however, is $\operatorname{Hom}(\bar{\mathcal{F}}(V), \bar{\mathcal{F}}(V)) \cong \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^2) \cong \operatorname{Mat}_{2\times 2}(\mathbb{C})$. Since the dimension of $\operatorname{Hom}(\bar{\mathcal{F}}(V), \bar{\mathcal{F}}(V))$ is greater than the dimension of $\operatorname{Hom}(V, V)$, the induced functor $\bar{\mathcal{F}}_{VV}$ is not surjective, and hence $\bar{\mathcal{F}}$ is not full.

Example 4.11. Next, consider the representation $V \in \operatorname{rep}(Q_\infty, R_\infty)$ given by $(V_i, x_i, \bar{x}_i) = (\mathbb{C}, \lambda, 1)$ for all $i \in \mathbb{Z}$, where $\lambda \in \mathbb{C}$. Suppose $V \cong \bar{\mathcal{F}}(U)$ for some $U \in \operatorname{rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$. Recall that the vertical maps y_{ij} of the representation U correspond to the leftward maps $\bar{x}_i$ of V. Since each V_i is one-dimensional, and each V_i maps to each V_{i-1} through the identity map, all nonzero U_{ij} must lie along the same column. Relabelling if necessary, we may assume it's the first column. Then we must have $U_{1j} \cong V_j$ and $y_{1j} \cong 1$. A similar argument shows that all nonzero U_{ij} must lie along the first *row*, with $U_{i1} \cong V_i$ and $x_{i1} \cong \lambda$. Clearly, no such U exists, and hence $\bar{\mathcal{F}}$ is not essentially surjective.

If I_∞ denotes the two sided ideal of $\mathbb{C}Q_\infty$ generated by the relations R_∞, then we have $\mathbb{C}Q_\infty/I_\infty = \mathcal{P}(Q^*_\infty)$, where Q^*_∞ is a subquiver of Q_∞ obtained by specifying an orientation. The functor $\bar{\mathcal{F}}$ provides an embedding of the category $\operatorname{rep}(\mathbb{C}Q_{\infty\times\infty}/J)$ in $\operatorname{rep}(\mathcal{P}(Q^*_\infty))$. The category $\operatorname{rep}(\mathcal{P}(Q^*_\infty))$ is well understood, and it is known that every finite dimensional representation of $\mathcal{P}(Q^*_\infty)$ is nilpotent, see ([8]). Thus the functor $\bar{\mathcal{F}}$ embeds the finite dimensional representations of $\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}$ inside $\Lambda_{Q^*_\infty}$.

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Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario K1N 6N5, Canada

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario K1N 6N5, Canada