University of Ottawa
Department of Mathematics and Statistics

MAT 4199/5145: Modern Group Theory
Professor: Alistair Savage

Final Exam – Solutions
December 2017

Surname _____ First Name _____

Student # _____

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Bonus	Total
Maximum	3	5	7	5	4	5	10	6	3	2		50
Grade												

QUESTION 1. **[3 points]**

(a) What is the content of the tableau

1	2	7	8
3	5	9	
4			
6			

?

Solution: The content of the given tableau is $(0, 1, -1, -2, 0, -3, 2, 3, 1)$.

(b) Write all the standard tableaux of shape $(3, 2)$.

Solution:

1	2	3
4	5	

, | | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | |

, | | | |
|---|---|---|
| 1 | 2 | 5 |
| 3 | 4 | |

, | | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 5 | |

, | | | |
|---|---|---|
| 1 | 3 | 5 |
| 2 | 4 | |

.

(c) What is the dimension of the irreducible representation $S^{(3,2)}$ of $\mathfrak{S}_5$?

Solution: Since $S^{(3,2)}$ has a basis enumerated by the standard tableaux of shape $(3, 2)$, we have

$$\dim S^{(3,2)} = 5.$$

QUESTION 2. **[5 points]**

(a) State Maschke's Theorem.

Solution: Every finite-dimensional representation of a finite group can be decomposed as a direct sum of irreducible representations. Furthermore, the decomposition can be chosen to be orthogonal (i.e. elements of distinct direct summands are orthogonal to each other).

(b) Suppose
$$\{1\} = G_1 \leq G_2 \leq \cdots \leq G_n$$
is a chain of subgroups. What does it mean for this chain to be *multiplicity free*?

Solution: The chain is multiplicity free if, for all $k = 2, \ldots, n$, and any irreducible representation ρ of G_k, the restriction $\mathrm{Res}^{G_k}_{G_{k-1}} \rho$ is multiplicity free (i.e. is a sum of irreducible representations, with each irreducible representation of G_k appearing with multiplicity at most one.)

(c) Suppose the chain
$$\{1\} = G_1 \leq G_2 \leq \cdots \leq G_n$$
is multiplicity free. Define the *Gelfand–Tsetlin algebra* associated with this chain.

Solution: The Gelfand–Tsetlin algebra is the subalgebra of $L(G_n)$ generated by the subalgebras
$$Z(L(G_1)), Z(L(G_2)), \ldots, Z(L(G_n)).$$

(d) Prove that the Gelfand–Tsetlin algebra is commutative.

Solution: It suffices to prove that, for all $1 \leq i \leq j \leq n$, elements of $Z(L(G_i))$ commute with elements of $Z(L(G_j))$. Let $a \in Z(L(G_i))$ and $b \in Z(L(G_j))$. Then
$$a \in Z(L(G_i)) \subseteq L(G_i) \subseteq L(G_j).$$
Since $b \in Z(L(G_j))$, it follows that $ab = ba$, as desired.

QUESTION 3. **[7 points]** Let G be a finite group.

(a) Define the *left regular representation* λ of G.

Solution:

$$\big(\lambda(g)f\big)(h) = f(g^{-1}h), \quad \text{for all } g,h \in G,\ f \in L(G).$$

(b) Define the *right regular representation* ρ of G.

Solution:

$$\big(\rho(g)f\big)(h) = f(hg), \quad \text{for all } g,h \in G,\ f \in L(G).$$

(c) Prove that, for all $g \in G$ and $f \in L(G)$, we have

$$\delta_g * f = \lambda(g)f.$$

Solution: For all $g,h \in G$ and $f \in L(G)$, we have

$$(\delta_g * f)(h) = \sum_{a\in G} \delta_g(a) f_2(a^{-1}h) = f_2(g^{-1}h) = (\lambda(g)f)(h).$$

(d) Prove that, for all $g \in G$ and $f \in L(G)$, we have

$$f * \delta_g = \rho(g^{-1})f.$$

Solution: For all $g, h \in G$ and $f \in L(G)$, we have

$$(f * \delta_g)(h) = \sum_{a \in G} f(ha^{-1})\delta_g(a) = f(hg^{-1}) = (\rho(g^{-1})f)(h).$$

(e) Prove that the center $Z(L(G))$ of the group algebra of G consists precisely of the class functions on G.

Solution: We have

$$\begin{aligned} f \in Z(L(G)) &\iff \delta_g * f = f * \delta_g, \quad \text{for all } g \in G \\ &\iff \lambda(g)f = \rho(g^{-1})f, \quad \text{for all } g \in G \\ &\iff f(g^{-1}h) = f(hg^{-1}), \quad \text{for all } g, h \in G \\ &\iff f \text{ is central.} \end{aligned}$$

QUESTION 4. **[5 points]** Let (σ, V) be a representation of a finite group G.

(a) Define what it means for $v \in V$ to be *cyclic*.

Solution: The vector v is cyclic if

$$\langle \sigma(g)v : g \in G \rangle = V.$$

(b) Prove that if σ is irreducible, then every nonzero $v \in V$ is cyclic.

Solution: Suppose $v \in V$, $v \neq 0$. The space

$$\langle \sigma(g)v : g \in G \rangle$$

is a nonzero σ-invariant subspace of V. If σ is irreducible, this must be equal to V.

(c) Give an example of a representation (σ, V) of a finite group G and a vector $v \in V$, such that σ is *reducible* and v is a cyclic vector. (Remember to justify your answer.) *Hint*: Try $G = \mathfrak{S}_2$.

Solution: Consider $G = \mathfrak{S}_2$ and the direct sum of the trivial and sign representations

$$V := S^{(2)} \oplus S^{(1,1)}.$$

Let v_1 be a nonzero vector in $S^{(2)}$, and let v_2 be a nonzero vector in $S^{(1,1)}$. Then define $v = v_1 + v_2$. We claim that v is cyclic. Indeed, let $U = \langle gv : g \in \mathfrak{S}_2 \rangle$. Then

$$v_1 - v_2 = s_1 v \in U.$$

Since $v_1 + v_2$ and $v_1 - v_2$ span V, we have $U = V$, proving the claim.

QUESTION 5. **[4 points]** Suppose that (σ, V) is a unitary representation of a finite group G, with character χ^σ.

(a) What is $\chi^\sigma(1_G)$, where 1_G is the identity element of the group G?

Solution: We have

$$\chi^\sigma(1_G) = \mathrm{tr}(\sigma(1_G)) = \mathrm{tr}(I_V) = \dim V.$$

(b) Prove that $\chi^\sigma\left(g^{-1}\right) = \overline{\chi^\sigma(g)}$ for all $g \in G$.

Solution: Let $\{v_i : 1 \leq i \leq d\}$ be an orthonormal basis for V. For $g \in G$, we have

$$\chi^\sigma\left(g^{-1}\right) = \sum_{i=1}^{d} \left\langle \sigma\left(g^{-1}\right) v_i, v_i \right\rangle = \sum_{i=1}^{d} \langle v_i, \sigma(g) v_i \rangle = \sum_{i=1}^{d} \overline{\langle \sigma(g) v_i, v_i \rangle} = \overline{\chi^\sigma(g)}.$$

QUESTION 6. **[5 points]** Let G be a finite group.

(a) Suppose that ρ is an irreducible representation of G. What is

$$\dim \mathrm{Hom}_G(\rho, \lambda),$$

where λ is the left regular representation of G?

Solution: We have that $\dim \mathrm{Hom}_G(\rho, \lambda)$ is multiplicity of ρ in the left regular representation, which is $\dim \rho$.

(b) Suppose that H is a subgroup of G, that ρ_G is an irreducible representation of G, and that ρ_H is an irreducible representation of H. Furthermore, suppose that the multiplicity of ρ_H in $\mathrm{Res}^G_H \rho_G$ is zero. What is the multiplicity of ρ_G in $\mathrm{Ind}^G_H \rho_H$?

Solution: By Frobenius reciprocity, we have

$$\begin{aligned}\text{multiplicity of } \rho_G \text{ in } \mathrm{Ind}^G_H \rho_H &= \dim_G(\rho_G, \mathrm{Ind}^G_H \rho_H) \\ &= \dim_H(\mathrm{Res}^G_H \rho_G, \rho_H) \\ &= \text{multiplicity of } \rho_H \text{ in } \mathrm{Res}^G_H \rho_G \\ &= 0.\end{aligned}$$

(c) Suppose that ρ is an irreducible representation of G and that σ is a representation (not necessarily irreducible) of G. Prove that

$$\dim \mathrm{Hom}_G(\rho, \sigma) \leq \frac{\dim \sigma}{\dim \rho}.$$

Solution: Recall that $\dim \mathrm{Hom}_G(\rho, \sigma)$ is the multiplicity of ρ in σ. Thus,

$$(\dim \rho)\,(\dim \mathrm{Hom}_G(\rho, \sigma)) \leq \dim \sigma,$$

giving the desired inequality.

QUESTION 7. **[10 points]**

(a) Write the representation

$$\operatorname{Res}^{\mathfrak{S}_5}_{\mathfrak{S}_3} S^{(3,1,1)}$$

as a sum of irreducible representations S^μ for appropriate partitions $\mu \vdash 3$.

Solution: We have

$$\begin{aligned}\operatorname{Res}^{\mathfrak{S}_5}_{\mathfrak{S}_3} S^{(3,1,1)} &= \operatorname{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_3} \operatorname{Res}^{\mathfrak{S}_5}_{\mathfrak{S}_4} S^{(3,1,1)} \\ &= \operatorname{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_3} \left(S^{(2,1,1)} \oplus S^{(3,1)}\right) \\ &= \left(\operatorname{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_3} S^{(2,1,1)}\right) \oplus \left(\operatorname{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_3} S^{(3,1)}\right) \\ &= S^{(1,1,1)} \oplus S^{(2,1)} \oplus S^{(2,1)} \oplus S^{(3)}.\end{aligned}$$

(b) Suppose $n \geq 1$. For which $\lambda \vdash n$ is the restriction $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ multiplicity free for *all* $k \in \{1, 2, \ldots, n\}$?

Solution: Considering $k = 1$, we have that $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_1} S^\lambda$ is irreducible if and only if it is one-dimensional. Clearly, if S^λ is one-dimensional, then $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ is multiplicity free for all values of k. Thus, it is precisely the λ corresponding to one-dimensional representations that have the desired property. So $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ is multiplicity free if and only if $\lambda = (n)$ or $\lambda = (1, 1, \ldots, 1)$.

(c) Suppose $n \geq 3$. For which $\lambda \vdash n$ is the restriction $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}} S^\lambda$ multiplicity free?

Solution: We have

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}} S^\lambda = \operatorname{Res}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}} \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda = \operatorname{Res}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}} \left(\bigoplus_{\substack{\mu \vdash n-1 \\ \lambda \to \mu}} S^\mu \right) = \bigoplus_{\substack{\nu \vdash n-2 \\ \mu \to \nu}} \bigoplus_{\substack{\mu \vdash n-1 \\ \lambda \to \mu}} S^\nu.$$

Thus, $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}} S^\lambda$ is multiplicity free if and only if, for every $\nu \vdash n-2$, there is at most one path from λ to ν in the Young graph. This is true if and only if the Young diagram corresponding to λ is a square, i.e. if $\lambda = (k, k, \ldots, k)$ for some $k \geq 1$. Indeed, if $\lambda = (k, k, \ldots, k)$, then the only removable box is the bottom-right one. Then removing a second box yields a different Young diagram for each choice of box. Conversely, if λ is not a square, then it has at least two removable boxes (the last box in the bottom row and the last box in any other row strictly longer than the row below it), which can removed in either order.

(d) Give an example of an irreducible representation V of $\mathfrak{S}_n$ (for some n) with $\dim V \geq 2$ such that $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V$ is irreducible. (Remember to justify your answer.)

Solution: It suffices to find a partition λ such that there are more than one standard tableaux of shape λ but such that λ has only one removable box. One example is $\lambda = (2, 2)$, which corresponds to the Young diagram

.

There are 2 standard tableaux of shape λ:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} .$$

Thus $\dim S^\lambda = 2$. In addition, $\operatorname{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_3} S^\lambda = S^{(2,1)}$ is irreducible.

QUESTION 8. **[6 points]** For $n \geq 1$, let C_n denote the cyclic group of order n, with generator a.

(a) Describe the irreducible representations of C_n.

Solution: The irreducible representations are χ_j, $0 \leq j \leq n-1$, defined by

$$\chi_k \colon C_n \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times, \quad \chi_j(a^k) = \exp\left(k\frac{2\pi j\sqrt{-1}}{n}\right), \quad k \in \mathbb{Z}.$$

(b) Recall that, if $k \geq 1$ divides n, then we can naturally identify C_k with the subgroup of C_n generated by $a^{n/k}$. Prove that

$$\{1\} \leq C_2 \leq C_6$$

is a multiplicity free chain.

Solution: Since all the irreducible representations of C_6 are one-dimensional, they remain irreducible after restriction to C_2. Similarly, the irreducible representations of C_2 remain irreducible after restriction to $\{1\}$.

(c) Draw the branching graph for the chain $\{1\} \leq C_2 \leq C_6$.

Solution: As noted above, we identify C_2 with the subgroup of C_6 generated by a^3. Let

$$\chi_j^{(6)}, \quad 0 \leq j \leq 5,$$

denote the irreducible representations of C_6 defined above, and let

$$\chi_j^{(2)}, \quad 0 \leq j \leq 1,$$

denote the irreducible representations of C_2.

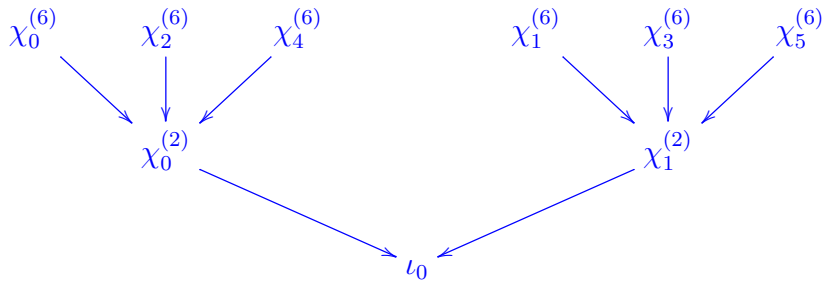
Suppose $0 \leq j \leq n-1$. Then,

$$\chi_j^{(6)}(a^3) = \exp\left(\frac{2\pi j\sqrt{-1}}{2}\right).$$

Hence

$$\mathrm{Res}^{C_6}_{C_2}\chi^{(6)} = \begin{cases} \chi_0^{(2)} & \text{if } j \text{ is even,} \\ \chi_1^{(2)} & \text{if } j \text{ is odd.} \end{cases}$$

Hence the branching graph is



where ι_0 denotes the trivial representation of $\{1\}$.

QUESTION 9. **[3 points]** Fix $n \geq 1$.

(a) Define the *Young–Jucys–Murphy* elements of $L(\mathfrak{S}_n)$.

Solution:

$$X_1 = 0, \quad X_k = \sum_{i=1}^{k-1} (i,k), \ k = 2, \ldots, n.$$

(b) Fix $k \in \{1, 2, \ldots, n\}$. What is the largest possible eigenvalue for the action of the Young–Jucys-Murphy element X_k on representations of $\mathfrak{S}_n$? What is the smallest possible eigenvalue? More precisely, what are the maximum and minimum of the set

$$\{\lambda \in \mathbb{Z} : X_k v = \lambda v \text{ for some nonzero } v \text{ in a representation of } \mathfrak{S}_n\}?$$

Solution: In the Young basis, the eigenvalues of X_k correspond to the content of the box labelled k in standard tableau. If T is a standard tableau, the column number of the box labelled k is between 1 and k. Similarly, the row number is between 1 and k. Thus, the content of the box is between $1-k$ and $k-1$. These bounds are attained in the sign and trivial representations, respectively.

QUESTION 10. **[2 points]**

(a) State Olshanskii's Theorem.

Solution: The centralizer algebra $Z(\ell, k) = \mathcal{C}(\mathfrak{S}_{\ell+k}, \mathfrak{S}_\ell)$ is generated by the Young–Jucys–Murphy elements $X_{\ell+1}, X_{\ell+2}, \ldots, X_{\ell+k}$, the subgroup $\mathfrak{S}_k$, and the center $Z(\ell)$ of $\mathfrak{S}_\ell$.

(b) Give Young's orthogonal form for the action of simple transpositions.

Solution: In the orthonormal basis $\{w_T : T \in \mathrm{Tab}(n)\}$ we have

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{s_j T},$$

where r is the axial distance from $j+1$ to j in T.

BONUS QUESTION **[3 points available]** Let

$$e = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi \in L(\mathfrak{S}_n).$$

Prove that e acts as the identity on the trivial one-dimensional representation of $\mathfrak{S}_n$ and as zero on any nontrivial irreducible representation.

Solution: Let (ρ, V) be an irreducible representation of $\mathfrak{S}_n$. If ρ is the trivial representation, then, for all $v \in V$, we have

$$ev = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} v = v.$$

Now suppose that ρ is not the trivial representation and let $v \in V$. If $ev \neq 0$, then it spans a one-dimensional σ-invariant subspace of V. Since V is irreducible, this implies that V is one-dimensional; hence ρ must be the sign representation. Let $1 \leq i \leq n-1$. Then we have

$$ev = \frac{1}{2n!} \sum_{\pi \in \mathfrak{S}_n} (\pi + \pi s_i)v = \frac{1}{2n!} \sum_{\pi \in \mathfrak{S}_n} (\pi v - \pi v) = 0.$$