uOttawa

Department of Mathematics
and Statistics

Département de mathématiques
et statistique

Analysis II – Mat 3120

Final Exam — Summer 2014

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Time: 3 hours.
This is a closed book exam. No electronic devices are allowed.
Attempt BOTH questions 1–2. Each question is worth 14 marks.
Within every question, sub-questions become progressively more difficult.
Total number of marks: 28.
The bonux question is worth 3 more marks.

(1) (a) Give defintions of a connected metric space; of a path-connected metric space. **[1 mark]**

(b) Give the definition of a real normed space. **[1 mark]**

(c) Let $x_n \to x$, $y_n \to y$ be two convergent sequences in a normed space E, and let $t, s \in \mathbb{R}$ be arbitrary scalars. Show that

$$tx_n + sy_n \to tx + sy.$$

[2 marks]

(d) A subset X of a real vector space E is called *convex* if for every $x, y \in X$ and each $t \in [0,1]$, one has

$$(1-t)x + ty \in X.$$

Prove that every open ball in a normed space is convex. **[2 marks]**

(e) Prove that every convex subset of a normed space is path-connected. **[2 marks]**

(f) Give an example of a convex subset X of a normed space E such that X is everywhere dense in E, and $X \neq E$. (You can refer to the results of the course if you wish.) **[2 marks]**

(g) Let X be a convex subset of a normed space E. Prove that the closure of X in E is convex. **[2 marks]**

(h) Prove that the interior of a convex subset of a normed space is convex. **[2 marks]**

[Continued on the next page....]

(2) (a) Give the definition of a compact set. **[1 mark]**

(b) Show that the set of points of a convergent sequence in a metric space, together with the limit of the sequence, forms a compact set. **[2 marks]**

(c) Give the definition of the normed space ℓ^2. **[1 mark]**

(d) Denote π_n the n-th coordinate projection from ℓ^2 to $\mathbb{R}$, sending a sequence x to its n-th coordinate x_n. Prove that π_n is continuous. **[2 marks]**

(e) Let $K \subseteq \ell^2$ be a compact subset. Denote $\kappa_n = \max\{|\pi_n(x)| : x \in K\}$. Prove that the real number sequence $(\kappa_n)_{n=1}^{\infty}$ converges to zero. (*Hint:* you may wish to use the fact that K is totally bounded.) **[2 marks]**

(f) Give an example showing that the sequence (κ_n) from the item (2e) need not be square summable. **[2 marks]**

(g) Suppose K is a closed subset of ℓ^2 with the property that the sequence (κ_n) defined as in the item (2e) converges to zero. Can we conclude that K is compact? Explain. **[2 marks]**

(h) Same question, assuming the sequence (κ_n) is square summable. **[2 marks]**

(3) (*Bonus question*) Given a subset X of a real vector space E, the *convex hull* of X, denoted $\operatorname{conv}(X)$, consists of all *convex linear combinations* of elements of X:

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N},\ x_i \in X,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

(It is easily seen to be the smallest convex subset of E containing X.)

Assume now that X is a totally bounded subset of a normed space E. Prove that $\operatorname{conv}(X)$ is also totally bounded.

Conclude that if X is compact and E is a Banach space, then the *closed convex hull* of X in E (the closure of $\operatorname{conv}(X)$) is compact. **[3 marks]**

[End of the exam questions]