[9] **1.** a) Let V and W be vector spaces over the same field $\mathbb{F}$, and let T be a function from V to W.

State what it means for T to be linear.

b) Are the following linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$? Either prove they satisfy the relevant conditions or show that they do not.

$$T_1(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} \qquad T_2(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x^2 \\ 3y^2 \end{bmatrix}$$

c) Let $R : V \to V$ and $S : V \to V$ be linear maps where V is some vector space.

Show that $R \circ S$ is also linear.

[8] **2.** a) Let $R : V \to W$ be a linear map, and X a subset of W.

Define $\ker(R)$ and $R^{-1}(X)$. Does X need to be a subspace of W?

b) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear mapping defined by $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y+z \\ x+y-z \end{bmatrix}$ (you do not need to show that T is linear). Let $X = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : a = 3b \right\}$.

Find $\ker(T)$ and $T^{-1}(X)$. Give your answers in the form of conditions.

[4] **3.** Let V be a vector space and M a subspace of V. Consider the quotient space V/M.

a) Give conditions on $x, y \in V$ such that $x + M = y + M$.

b) For which value(s) of $z \in V$ is $z + M$ the zero vector of V/M? If there are no such values then explain why not; otherwise determine them all.

[+4] **4.** a) (bonus) Let M and N be subspaces of some vector space V such that $V = M + N$. Prove that if $M \cap N = \{\theta\}$ then for each vector $x \in V$ there exist *unique* vectors $m \in M$ and $n \in N$ such that $x = m + n$.

(hint: assume $m_1 + n_1 = m_2 + n_2$ and find something in the intersection.)

b) (bonus) What is the kernel of a linear transformation useful for? State something we used it for in this course. You do not need to prove that your "use" is correct, merely state it clearly.

[7] **5.** a) State the First Isomorphism Theorem. Be sure to include all hypotheses.

b) Use the First Isomorphism Theorem to show that $(M \oplus N)/M \cong N$, where M and N are subspaces of some vector space V with $M \cap N = \{\theta\}$.

(hint: start with the quotient map from V to V/M and restrict to N.)

[5] **6.** a) Give the definition of what it means for a set of vectors $x_1, x_2, \cdots, x_m$ to be linearly

dependent. **Do not** answer "they are not linearly independent".

b) Let $T : V \to W$ be an injective linear map, and let $x_1, x_2, \cdots, x_n$ be vectors such that $Tx_1, Tx_2, \cdots, Tx_n$ are linearly dependent in W. Show that $x_1, x_2, \cdots, x_n$ are linearly dependent in V.